Arch. Math. 68 (1997) 441–449
0003-889X/97/060441-09 $ 3.30/0

Archiv der Mathematik

Conjugacy separability of certain torsion groups

By

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Abstract. It is shown that certain torsion groups studied by Grigorchuk and by Gupta and Sidki are conjugacy separable.

1. Introduction. In [1, 2, 3], R. I. Grigorchuk introduced and studied some remarkable new examples of finitely generated torsion groups. These groups are residually finite and amenable, and they fall into $2^{\aleph_0}$ isomorphism classes of p-groups for each prime p. Grigorchuk solved a problem of Milnor by showing that their growth is faster than polynomial growth and slower than exponential growth, and he gave in [2] an explicit criterion describing which of these groups have soluble word problem. However, the conjugacy problem for these groups seems to remain open. We are grateful to Professor Grigorchuk for pointing this out to us and for a number of helpful suggestions.

We recall that a group G is *conjugacy separable* if for every pair g, h of elements of G which are not conjugate there is a finite quotient group of G in which the images of g, h fail to be conjugate; equivalently, G is conjugacy separable if (i) G is residually finite, so that it embeds naturally in its profinite completion $\widehat{G}$, and (ii) elements of G which are conjugate in $\widehat{G}$ are also conjugate in G. A well-known argument of McKinsey [5] shows that if G is a finitely generated recursively presented group and if G is conjugacy separable then the conjugacy problem for G is soluble.

We shall prove that for p odd the p-groups of Grigorchuk described above are conjugacy separable. It follows from this that if G is one of these p-groups then G has soluble conjugacy problem if and only if it has soluble word problem. Further remarkable examples of finitely generated residually finite p-torsion groups have been discovered and studied by N. Gupta and S. Sidki [4, 6], and we shall show that for p odd they too are conjugacy separable. Since these latter groups are recursively presented, it follows that they have soluble conjugacy problem.

The groups mentioned above have a number of common features. Each of them can be embedded as a subgroup of finite index in the wreath product of a similar group and a finite cyclic group, and there is a length function which behaves well with respect to the embedding. We shall not give here the details of the constructions of these groups; instead we shall prove a result sufficiently general to handle all of them simultaneously. As a by-product, we shall prove that the standard wreath product of a conjugacy separable group and a finite group is conjugacy separable.

Unfortunately the 2-groups of Grigorchuk and of Gupta and Sidki are not covered by our result. Although they can probably be handled using similar techniques (especially the very explicitly described group in [1]), we believe that the arguments required would be considerably more complicated.

This paper was written while the second author was visiting the University of Birmingham. He would like to thank the School of Mathematics and Statistics for its support and warm hospitality.

2. Wreath products and the main theorem. First we establish our notation for wreath products. Let H, K be groups, and write B for the group of all functions of finite support from K to H with pointwise multiplication. Let K act on the right on B as follows:

$$b^k(x) = b(x\,k^{-1}) \quad \text{for} \quad k, x \in K, b \in B.$$

The standard (restricted) wreath product $W = H \operatorname{wr} K$ is the split extension of B by K, and B is the base group of W. If K is a finite cyclic group of order m generated by t we shall find it convenient to denote the element $b \in B$ by the vector $(b(1), b(t), \ldots, b(t^{m-1}))$; thus in this notation we have

$$t^{-1}(h_1, \ldots, h_m)t = (h_m, h_1, \ldots, h_{m-1})$$

for all $(h_1, \ldots, h_m) \in B$.

Lemma 1. *Let $g_1 = b_1k, g_2 = b_2k$ for some $k \in K, b_1, b_2 \in B$ and suppose that k has finite order m. If g_1^m, g_2^m are conjugate in B then $g_1^b = g_2$ for some $b \in B$.*

Proof. Suppose that $(g_1^m)^d = g_2^m$, where $d \in B$. We have

$$g_1^m = b_1k \cdots b_1k = b_1 b_1^{k^{-1}} \cdots b_1^{k^{-(m-1)}}$$

since $k^m = 1$. Thus for each $x \in K$ we have

$$g_1^m(x) = b_1(x)b_1(x\,k) \cdots b_1(x\,k^{m-1}).$$

Similarly

$$g_2^m(x) = b_2(x)b_2(x\,k) \cdots b_2(x\,k^{m-1}),$$

and so

$$(*) \qquad d^{-1}(x)b_1(x)b_1(x\,k) \cdots b_1(x\,k^{m-1})d(x) = b_2(x)b_2(x\,k) \cdots b_2(x\,k^{m-1}),$$

for each $x \in K$. Let Y be a transversal to the cosets $x\langle k\rangle$ of $\langle k\rangle$ in K. For each $y \in Y$ we set $b(y) = d(y)$ and then we define $b(yk^i)$ for $i = 1, \ldots, m-1$ by

$$b(y\,k^i) = b_1^{-1}(y\,k^{i-1})b(y\,k^{i-1})b_2(y\,k^{i-1}).$$

Thus $b \in B$, and from $(*)$ we have

$$b(y) = b_1^{-1}(y\,k^{m-1})b(y\,k^{m-1})b_2(y\,k^{m-1})$$

for each $y \in Y$. Therefore $b(x\,k) = b_1^{-1}(x)b(x)b_2(x)$ for all $x \in K$. We conclude that $b^{k^{-1}} = b_1^{-1}b\,b_2$, and hence that $(b_1k)^b = b_2k$, as required.

Proposition. *If H is conjugacy separable and K is finite then W is conjugacy separable.*

Proof. We denote by $\widehat{U}$ the profinite completion of a group U. Since K is finite the inclusion map from $H \operatorname{wr} K$ to $\widehat{H} \operatorname{wr} K$ extends to an isomorphism from $\widehat{W}$ to $\widehat{H} \operatorname{wr} K$. Let C be the base group of $\widehat{H} \operatorname{wr} K$. Thus we have $\widehat{H} \operatorname{wr} K = CK$.

Let g_1, g_2 be elements of W conjugate in $\widehat{H} \operatorname{wr} K$. We must show that g_1, g_2 are conjugate in W. We have $g_1^{ck'} = g_2$ for some $c \in C, k' \in K$, and replacing g_2 by $g_2^{k'^{-1}}$ we may suppose that $g_1^c = g_2$ with $c \in C$. Thus the images of g_1, g_2 under the natural map from $\widehat{H} \operatorname{wr} K$ to K are equal, and we can write $g_1 = b_1 k, g_2 = b_2 k$ with $b_1, b_2 \in B$ and $k \in K$. Let k have order m. By Lemma 1, g_1, g_2 are certainly conjugate in W if g_1^m, g_2^m are conjugate in B. Since $(g_1^m)^c = g_2^m$, for each $x \in K$ the elements $g_1^m(x), g_2^m(x)$ are conjugate in $\widehat{H}$, and hence are conjugate in H. Therefore g_1^m, g_2^m are conjugate in B and the result follows.

If x is an element of a group G and Y is a subset of G, we write x^Y to denote the set $\{x^y \mid y \in Y\}$. Thus if G is residually finite and we regard G as embedded in its profinite completion $\widehat{G}$, then G is conjugacy separable if and only if $g^{\hat{G}} \cap G = g^G$ for each $g \in G$.

Theorem. *Let $\mathscr{X}$ be a class of residually finite groups. Suppose that for every group G in $\mathscr{X}$ the abelianization G/G' is finite and there exist an integer m, an element t of G of order m, a group G_0 in $\mathscr{X}$ and an embedding φ of G in the standard wreath product $W = G_0 \operatorname{wr} \langle t \rangle$, such that $\varphi(t) = t$ and such that the commutator subgroup B' of the base group B of W is contained in $\varphi(G')$. Suppose also that there is a positive integer N and a function $\varphi : G \to \mathbb{N}$ assigning a 'length' $\ell(g)$ to each element g of G such that the following conditions hold:*

(i) *if $g \in G$ and $\ell(g) \leqq N$ then $g^{\overline{G'}} \cap G = g^{G'}$, where $\overline{G'}$ denotes the closure of G' in $\widehat{G}$;*
(ii) *if $g \in G$, $\ell(g) > N$ and $\varphi(g)$ is in B, say $\varphi(g) = (g_1, \ldots, g_m)$, then $\ell(g_i) < \ell(g)$ for each i;*
(iii) *if $g \in G$, $\ell(g) > N$ and $\varphi(g) \notin B$, say $\varphi(g) = (g_1, \ldots, g_m)t^r$ with $t^r \neq 1$, then for $i = 1, \ldots, m-1$ we have $\ell(g_i g_{i+r} \cdots g_{i+(k-1)r}) < \ell(g)$, where k is the order of t^r and indices are computed modulo m.*

Then all groups in $\mathscr{X}$ are conjugacy separable.

Proof. We begin with an elementary observation. Let G be in $\mathscr{X}$ and $g \in G$, and suppose that $g^{\overline{G'}} \cap G = g^{G'}$. Since $\widehat{G}/\overline{G'} \cong G/G'$, the group $\widehat{G}$ is the union of finitely many cosets $\overline{G'}u$ with $u \in G$, and for each of them we have

$$g^{\overline{G'}u} \cap G = (g^{\overline{G'}} \cap G)^u \subseteqq g^{G'u},$$

and thus $g^{\widehat{G}} \cap G = g^G$.

To prove the theorem, it is therefore enough to prove that if G is in $\mathscr{X}$ and $g \in G$ has length n then $g^{\overline{G'}} \cap G = g^{G'}$. We shall do this simultaneously for all groups G in $\mathscr{X}$, by induction on n. By (i), the result holds for $n \leqq N$ and so we may assume $n > N$.

Fix a group G in $\mathscr{X}$, and an element $g \in G$ with length n. Let m, t, G_0 and φ be as given by the hypotheses of the theorem. Write B for the base group of the wreath product $W = G_0 \operatorname{wr} \langle t \rangle$. Since $|\langle t \rangle|$ and $|W : \varphi(G)|$ are finite (the latter being so since $B' \leqq \varphi(G)$ and since G_0/G_0' is finite), the profinite completions of W, $\varphi(G)$ can be identified with $\widehat{G_0} \operatorname{wr} \langle t \rangle$ and the closure of $\varphi(G)$ in $\widehat{G_0} \operatorname{wr} \langle t \rangle$. Moreover, since W/B is abelian we have $\varphi(G') \leqq B$, and so $\overline{\varphi(G')}$ has finite index in the base group $\overline{B}$ of $\widehat{G_0} \operatorname{wr} \langle t \rangle$.

Let $h \in g^{\overline{G'}} \cap G$ and choose $\delta \in \overline{\varphi(G')}$ with $\varphi(g)^{\delta} = \varphi(h)$. Write $\varphi(g) = v t^r, \varphi(h) = w t^s$ with v, w in B. Since the images of $\varphi(g), \varphi(h)$ under the natural map from the wreath product to $\langle t \rangle$ are conjugate, we have $s = r$.

We shall prove the following two assertions:

(a) $\varphi(g), \varphi(h)$ are conjugate under an element of B, and
(b) $C_{\overline{B}}(\varphi(g))\overline{B'} = C_B(\varphi(g))\overline{B'}$.

Case (i). Suppose that $t^r = 1$, so that $v = \varphi(g), w = \varphi(h)$. Write $v = (g_1, \ldots, g_m)$. By hypothesis (ii), we have $\ell(g_i) < n$ for each i. Since $\varphi(g)^{\delta} = \varphi(h)$ and $\delta \in \overline{\varphi(G')} \leqq \overline{B}$, the respective components of the vectors $\varphi(g), \varphi(h)$ are conjugate in $\widehat{G_0}$ and so, by induction and the observation in the first paragraph, are conjugate in G_0. It follows that $\varphi(g), \varphi(h)$ are conjugate in B.

Now let $\beta \in C_{\overline{B}}(\varphi(g))$. Since $|B : B'|$ is finite we have $\overline{B} = B\overline{B'}$, and so we can write $\beta = b_1 b_2^{-1}$ with $b_1 \in B, b_2 \in \overline{B'}$. Then $\varphi(g)^{b_2} = \varphi(g)^{b_1}$. Thus the components of $\varphi(g)^{b_2}$ are in G_0 and the components of b_2 are in $\overline{G_0'}$, and so our induction hypothesis yields an element $b \in B'$ with $\varphi(g)^b = \varphi(g)^{b_2} = \varphi(g)^{b_1}$. Therefore $b_1 b^{-1} \in C_B(\varphi(g))$ and $b\, b_2^{-1} \in \overline{B'}$, so that $\beta = (b_1 b^{-1})(b\, b_2^{-1}) \in C_B(\varphi(g))\overline{B'}$, and assertion (b) follows.

Case (ii). Suppose that $t^r \neq 1$ and let t^r have order k. Write $x = g^k, y = h^k, v = (g_1, \ldots, g_m)$, $\varphi(x) = (x_1, \ldots, x_m)$. We have

$$\varphi(x) = (v t^r)^k = v v^{t^{-r}} \cdots v^{t^{-r(k-1)}},$$

so that $x_i = g_i\, g_{i+r} \cdots g_{i+(k-1)r}$ for each i. By Hypothesis (iii), we have $\ell(x_i) < n$ for each i. Since $\varphi(g^k)^{\delta} = \varphi(h^k)$, we conclude just as in Case (i) that $\varphi(g^k), \varphi(h^k)$ are conjugate in B. It follows from Lemma 1 that $\varphi(g), \varphi(h)$ are conjugate under an element of B.

Let $\beta = (\beta_1, \ldots, \beta_m) \in \overline{B}$. We have $\beta \in C_{\bar{B}}(\varphi(g))$ if and only if $v \beta^{t^{-r}} = \beta v$, and this holds if and only if

$$\beta_{i+r} = g_i^{-1} \beta_i\, g_i \quad \text{for all} \quad i.$$

This in turn holds if and only if both of the following conditions hold:

(1) $$\beta_{i+jr} = g_{i+(j-1)r}^{-1}\, \beta_{i+(j-1)r}\, g_{i+(j-1)r} \quad \text{for} \quad 0 < i \leqq r \text{ and } 1 \leqq j < m/r$$

and

(2) $$g_i\, g_{i+r} \cdots g_{i+(k-1)r}\, \beta_i = \beta_i\, g_i\, g_{i+r} \cdots g_{i+(k-1)r} \quad \text{for} \quad 0 < i \leqq r.$$

However $x_i = g_i\, g_{i+r} \ldots g_{i+(k-1)r}$, so that Condition (2) above is simply the requirement that $\beta_i \in C_{\hat{G}_0}(x_i)$ for $0 < i \leqq r$, and we have $\ell(x_i) < n$ for each i. By induction and the argument of Case (i) for the components labelled $1, \ldots, r$, we find elements $c_1, \ldots, c_r$ with $c_i \in C_B(x_i)$ and with β_i, c_i in the same coset of $\overline{G_0'}$ in $\widehat{G_0}$ for $1 \leqq i < r$. Now define

$$c_{i+jr} = g_{i+(j-1)r}^{-1}\, c_{i+(j-1)r}\, g_{i+(j-1)r} \quad \text{for} \quad 1 \leqq i \leqq r \text{ and } 1 \leqq j < m/r.$$

It follows from Conditions (1) and (2) that $c = (c_1, \ldots, c_m) \in C_B(\varphi(g))$. Since $\overline{G_0'}$ is normal in $\widehat{G_0}$ it follows also that for each $i \leqq m$ the elements β_i, c_i are in the same coset of $\overline{G_0'}$. Therefore $\beta \in c\,\overline{B'} \leqq C_B(\varphi(g))\overline{B'}$, and assertion (b) follows.

Thus in each case the assertions (a), (b) hold. Let $\varphi(g)^b = \varphi(h)$ with $b \in B$. Since also $\varphi(g)^\delta = \varphi(h)$ with $\delta \in \overline{\varphi(G')}$, we have $b \in C_{\overline{B}}(\varphi(g))\overline{\varphi(G')}$. Now $\overline{B'}$ is open in $\overline{\varphi(G')}$ and $\varphi(G')$ is dense in $\overline{\varphi(G')}$, and so we have $\overline{\varphi(G')} = \overline{B'}\varphi(G')$. Thus by assertion (b) we have

$$b \in C_{\overline{B}}(\varphi(g))\overline{B'}\varphi(G') = C_B(\varphi(g))\overline{B'}\varphi(G').$$

Since the first and third factors here are contained in B and since clearly $\overline{B'} \cap B = B'$ it follows that

$$b \in C_B(\varphi(g))(\overline{B'} \cap B)\varphi(G') = C_B(\varphi(g))B'\varphi(G').$$

But $B' \leqq \varphi(G')$ from the hypotheses of the Theorem, and therefore we can write $b = c_1\varphi(d)$ with $c_1 \in C_B(\varphi(g))$ and $d \in G'$. Since $\varphi(h) = \varphi(g)^{\varphi(d)} = \varphi(g^d)$ we have $h = g^d$, and the result is proved.

3. Application to torsion groups. We now explain how the above Theorem can be applied to some p-torsion groups with $p \neq 2$ described by Grigorchuk and by Gupta and Sidki. The groups we consider here are

(i) the groups arising in Theorem 1 of Grigorchuk [3], i.e. the group G_ω for each map $\omega : \mathbb{N} \to \{0, 1, \ldots, p\}$ taking each of the values $0, 1, \ldots, p$ infinitely often, where $p > 2$, and

(ii) the groups treated in the Theorem of Gupta and Sidki [4].

We take for $\mathscr{X}$ the class containing all groups in (i), (ii) above. Each of these groups G is residually finite and is a p-group satisfying $G = \langle \Gamma, t \rangle$, where Γ is a finite elementary abelian p-group disjoint from G' and t has order p, for some odd prime p. We fix one of these groups $G = \langle \Gamma, t \rangle$. There is a group $G_0 = \langle \Gamma_0, t_0 \rangle \in \mathscr{X}$ such that t, t_0 have the same order, p, say, and such that there is an isomorphism $\beta : \Gamma \to \Gamma_0$. There are homomorphisms $\alpha_1, \ldots, \alpha_{p-1} : \Gamma \to \langle t_0 \rangle$, with $\alpha_2(\gamma) = \ldots = \alpha_{p-1}(\gamma) = 1$ for all $\gamma \in \Gamma$ for the examples of Grigorchuk, and with $\alpha_1(\gamma)\alpha_2(\gamma) = 1$ and $\alpha_3(\gamma) = \ldots = \alpha_{p-1}(\gamma) = 1$ for all $\gamma \in \Gamma$ for the examples of Gupta and Sidki; moreover $\alpha_1(\gamma_1) = t_0$ for some $\gamma_1 \in \Gamma$. There is a monomorphism φ from the group G to the wreath product $W = G_0 \operatorname{wr} \langle t \rangle$ satisfying $\varphi(t) = t$ and $\varphi(\gamma) = (\alpha_1(\gamma), \ldots, \alpha_{p-1}(\gamma), \beta(\gamma))$ for each $\gamma \in \Gamma$. All this can be read off without difficulty from the descriptions of these groups G in [3, 4]. Write $T = \langle t \rangle$ and write B for the base group of W.

Since G is a finitely generated torsion group, its abelianization is finite. This is one of the requirements in the Theorem; we check another in the following lemma.

Lemma 2. *$\varphi(G')$ contains B'.*

Proof. For the example of Gupta and Sidki [4] with $p = 3$ this follows from result 2.2.1 of Sidki [6]. In the other cases we have $\alpha_{p-1}(\gamma) = 1$ for all $\gamma \in \Gamma$. Let $\gamma \in \Gamma$, let $\gamma_1 \in \Gamma$ satisfy $\alpha_1(\gamma_1) = t_0$ and let $1 \leqq i \leqq p - 1$. Since Γ_0 is abelian we have

$$\begin{aligned}\varphi([\gamma_1^i, t^{-1}\gamma t]) &= [(\alpha_1(\gamma_1^i), \ldots, \alpha_{p-1}(\gamma_1^i), \beta(\gamma_1^i)), (\beta(\gamma), \alpha_1(\gamma), \ldots, \alpha_{p-1}(\gamma))] \\ &= ([t_0^i, \beta(\gamma)], 1, \ldots, 1).\end{aligned}$$

Since $\{[t_0^i, \delta] \mid 1 \leqq i \leqq p - 1, \delta \in \Gamma_0\}$ generates the derived group G_0' of G_0 and since $\varphi(G)$ is invariant under conjugation by t we conclude that $B' \leqq \varphi(G')$, as required.

It remains now to define a length function on G (and on all other groups in $\mathscr{X}$), such that hypotheses (i), (ii), (iii) of the theorem hold.

Denote the union of the conjugates of Γ under the elements of T by Γ^T. We have $\varphi^{-1}(B) \lhd G$ and so $\langle \Gamma^T \rangle \leqq \varphi^{-1}(B)$. If $g \in G$ then we may write $g = u\,t^r$ with $u \in \varphi^{-1}(B)$ and $0 \leqq r < p$. It follows that $\langle \Gamma^T \rangle = \varphi^{-1}(B)$, and that $G = \langle \Gamma^T \rangle \rtimes T$.

Now let $g \in G$. Then there are products of elements of $\Gamma \cup T$ equal to g; let $l_1(g)$ be the smallest number of elements of Γ appearing in such a product and let $o(g)$ be the order of g. We define

$$l(g) = l_1(g) + o(g).$$

Note that if $g \in \langle \Gamma^T \rangle$ then g is a product of $l_1(g)$ elements of Γ^T. We define length functions ℓ_1, ℓ in exactly the same way for all other groups in $\mathscr{X}$.

Lemma 3. *If $g \in \langle \Gamma^T \rangle$, say $\varphi(g) = (g_1, \ldots, g_p)$, then either $g \in \Gamma^T$ or $l(g_i) < l(g)$ for each i.*

Proof. Clearly $o(g_i) \leqq o(g)$ for each i. Let $n = l_1(g)$, and write $g = u_1 \ldots u_n$ with each $u_j \in \Gamma^T$. From the definition of φ, each g_i is a product of n elements from $\Gamma_0 \cup \langle t_0 \rangle$, and we can only have $l_1(g_i) = n$ if each of the factors belongs to Γ. However then we clearly have $l_1(g_i) = 1$, and the result follows.

Lemma 4. *If $g \in G \setminus \langle \Gamma^T \rangle$, say $\varphi(g) = (g_1, \ldots, g_p)t^r$ with $1 \leqq r < p$, then for each $k \leqq p$ we have $l(g_k\, g_{k+r} \cdots g_{k+(p-1)r}) < l(g)$.*

Proof. The element $\varphi(g)^p$ belongs to B and, by the calculation in the proof of Lemma 1, the element $d_k = g_k\, g_{k+r} \ldots g_{k+(p-1)r}$ appears as one of its components. Therefore $o(d_k) < o(g)$ and so it will suffice to show that $l_1(d_k) \leqq l_1(g)$.

Let $g = h\,t^r$, so that $\varphi(h) = (g_1, \ldots, g_p)$ and $l_1(g) = l_1(h)$. Set $n = l_1(g)$ and write $h = u_1 \ldots u_n$ with each $u_j \in \Gamma^T$.

Let π_i be the projection $b \mapsto b(t^{i-1})$ from B onto its ith direct factor. The element d_k is a product in some order of all of the elements $\pi_j(u_i)$ with $j \leqq p, i \leqq n$. From the definition of φ, each $\pi_j(u_i)$ belongs to $\Gamma_0 \cup \langle t_0 \rangle$, and moreover for each i there is a unique j with $\pi_j(u_i) \notin \langle t_0 \rangle$. It follows that $l_1(d_k) \leqq n$, as required.

The above two lemmas show that the Hypotheses (ii), (iii) of the Theorem are satisfied, with $N = p + 1$. We must now consider conjugates in $\widehat{G}$ of non-trivial elements g with $\ell(g) \leqq p + 1$. Because G is a p-group, these elements g have order p, and they are expressible in the form $t_1\, \gamma\, t_2$ with $\gamma \in \Gamma$ and $t_1, t_2 \in T$.

Lemma 5. (a) *Each element g of order p and length at most $p + 1$ is conjugate under an element of T to an element of $\Gamma \cup T$.*

(b) *Distinct elements of $\Gamma \cup T$ cannot be conjugate in $\widehat{G}$.*

Proof. (a) Since each element $t_1\, \gamma\, t_2$ is conjugate under an element of T to an element $\gamma\, t^r$, it will suffice to show that if $\gamma \in \Gamma$ and $1 \leqq r < p$ then $\gamma\, t^r$ has order p only if $\gamma = 1$. We have

$$\gamma\, \gamma^{k^{-r}} \cdots \gamma^{t^{-r(p-1)}} = 1.$$

Applying φ and taking last components, we have $\beta(\gamma) w_2 \cdots w_p = 1$, where $w_2, \ldots, w_p$ are the elements $\alpha_1(\gamma),\dots,\alpha_{p-1}(\gamma)$ in some order. Since the elements $\alpha_i(\gamma)$ lie in $\langle t_0\rangle$ and since $\Gamma_0\cap\langle t_0\rangle=1$, it follows that $\beta(\gamma)=1$ and hence that $\gamma=1$.

(b) Let ψ be the product of φ and the natural homomorphism from W to the finite group $F=G_0/G_0'\,\mathrm{wr}\,\langle t\rangle$. It will suffice to show that the images in F of the elements of $\Gamma\cup T$ are non-conjugate. Clearly neither the images of two distinct elements of T nor the images of a non-trivial element of T and an element of Γ can be conjugate in F, since such pairs of elements of F map to pairs of distinct elements under the homomorphism from the wreath product F to the abelian group T.

Now let γ_1,γ_2 be non-trivial elements of Γ. The last components of $\varphi(\gamma_1),\varphi(\gamma_2)$ are $\beta(\gamma_1),\beta(\gamma_2)$, and these elements are not in G_0', since from our hypotheses on groups in $\mathscr{X}$ the map β is injective and $G_0'\cap\Gamma_0=1$. Moreover, since G_0 is the semidirect product of the normal subgroup generated by Γ_0 and the subgroup $\langle t_0\rangle$, non-trivial elements of $\langle t_0\rangle$ cannot be congruent or conjugate modulo G_0' in G_0 to elements of Γ_0, and so $\beta(\gamma_1),\beta(\gamma_2)$ are not in $G_0'\langle t_0\rangle$. Thus the final components of $\psi(\gamma_1),\psi(\gamma_2)$ fail to lie in $G_0'\langle t_0\rangle/G_0'$. Referring to the definition of the map φ, we see that all other components of $\psi(\gamma_1),\psi(\gamma_2)$ do lie in the subgroup $G_0'\langle t_0\rangle/G_0'$. It follows from our remark above about conjugacy modulo G_0' that if $\psi(\gamma_1),\psi(\gamma_2)$ are conjugate in F under an element f then f must belong to the base group of F, and hence that $\psi(\gamma_1)=\psi(\gamma_2)$. Thus $\beta(\gamma_1\gamma_2^{-1})\in G_0'\cap\Gamma_0=1$, and $\gamma_1=\gamma_2$. Our result follows.

Lemma 6. (a) *If γ is a non-trivial element of Γ then $C_{\widehat{G}}(\gamma)\overline{G'}=C_G(\gamma)\overline{G'}$.*
(b) *If $g\in\Gamma\cup T$, then $C_{\overline{B}}(\varphi(g))\overline{B'}=C_B(\varphi(g))\overline{B'}$.*

Proof. (a) Clearly $\Gamma\overline{G'}\leqq C_G(\gamma)\overline{G'}\leqq C_{\widehat{G}}(\gamma)\overline{G'}$, and since $\widehat{G}/\Gamma\overline{G'}$ has order p it will suffice to prove that $t\notin C_{\widehat{G}}(\gamma)\overline{G'}$, or, equivalently, that $C_{\widehat{G}}(\gamma)t\cap\overline{G'}=\emptyset$.

We identify the profinite completion $\widehat{W}$ of W with $\widehat{G_0}\,\mathrm{wr}\,\langle t\rangle$ and write $\widehat{\varphi}$ for the isomorphism from $\widehat{G}$ to $\overline{\varphi(G)}$ extending φ. Suppose that δ is the image under $\widehat{\varphi}$ of an element of $C_{\widehat{G}}(\gamma)t\cap\overline{G'}$. Then

$$\delta\in\widehat{\varphi}(\overline{G'})=\overline{\varphi(G')}\leqq\overline{W'}\leqq\overline{B},$$

and so the first component of $\varphi(\gamma)^{\delta}$ is conjugate in $\widehat{G_0}$ to a (possibly trivial) element of $\langle t\rangle$. On the other hand, since $\delta\in\widehat{\varphi}(C_{\widehat{G}}(\gamma)t)\leqq C_{\widehat{W}}(\varphi(\gamma))t$, we have $\varphi(\gamma)^{\delta}=\varphi(\gamma)^{t}$, and the first component of this element is $\beta(\gamma)$. This leads to a contradiction to Lemma 5 applied to the group G_0, and our assertion follows.

(b) If $g\in T$ then

$$C_{\overline{B}}(g)=\{b\in\overline{B}\mid b(t^{i-1})=b(1)\quad\text{for}\quad 1\leqq i<p\}$$

and

$$C_B(g)=\{b\in B\mid b(t^{i-1})=b(1)\quad\text{for}\quad 1\leqq i<p\},$$

and since $\widehat{G_0}=\overline{G_0'}G_0$ we have

$$C_{\overline{B}}(\varphi(g))\overline{B'}=C_B(\varphi(g))\overline{B'}=\{b\in\overline{B}\mid b(t^{i-1})b(1)^{-1}\in\overline{G_0'}\text{ for }1\leqq i<p\}.$$

Let $g\in\Gamma$. Then $\varphi(g)=(\alpha_1(g),\dots,\beta(g))$, and, since $\overline{B}$ is isomorphic to the base group of the wreath product $\widehat{G_0}\,\mathrm{wr}\,\langle t\rangle$ we must show that

$$C_{\widehat{G_0}}(\beta(g))\overline{G_0'}=C_{G_0}(\beta(g))\overline{G_0'}$$

and

$$C_{\widehat{G_0}}(\alpha_i(g))\overline{G_0'} = C_{G_0}(\alpha_i(g))\overline{G_0'} \quad \text{for} \quad i < p.$$

The first of these requirements follows from part (a) of the lemma, applied to G_0 instead of G, and since $\alpha_i(g) \in T_0$ for each $i < p$ the second requirement follows by arguing as in the above paragraph, with G_0 replacing G.

Lemma 7. *If* $g \in \Gamma \cup T$ *then* $g^{\overline{G'}} \cap G = g^{G'}$.

Proof. Let $h \in g^{\overline{G'}} \cap G$. We shall argue by induction on the length n of h that h is conjugate to g under an element of G'. If $n \leqq p+1$ the result follows from Lemma 5 (a) and the remarks preceding it. The rest of the argument follows the strategy of the proof of our Theorem. We identify the profinite completions of W, $\varphi(G)$ with $\widehat{G_0}\,\mathrm{wr}\,\langle t\rangle$ and $\overline{\varphi(G)}$. Choose $\delta \in \overline{\varphi(G')}$ with $\varphi(g)^{\delta} = \varphi(h)$.

First we show that $\varphi(g), \varphi(h)$ are conjugate under an element of B.

Case (i). Suppose that $g \in \Gamma$, so that $\varphi(g) \in B$. Since the images of $\varphi(g), \varphi(h)$ in W/B are conjugate, we also have $\varphi(h) \in B$. Write $v = (g_1, \dots, g_m)$ and $w = (h_1, \dots, h_m)$. By Lemma 3 we have $\ell(h_i) < n$ for each i. Since $\varphi(g)^{\delta} = \varphi(h)$ with $\delta \in \overline{B}$, the respective components of $\varphi(g), \varphi(h)$ are conjugate in $\widehat{G_0}$. However, the components of $\varphi(g)$ lie in $\Gamma_0 \cup \langle t_0\rangle$, and so, by induction and the observation at the start of the proof of the Theorem, the respective components of $\varphi(g), \varphi(h)$ are conjugate in G_0. It follows that $\varphi(g), \varphi(h)$ are conjugate under an element of B.

Case (ii). Suppose that $g = t^r$ for some integer r. Since the images of $\varphi(g), \varphi(h)$ under the map from the wreath product W to T are conjugate, $\varphi(g), \varphi(h)$ lie in the same coset of B in W. Both $\varphi(g), \varphi(h)$ have order p, and so these elements are conjugate under an element of B by Lemma 1.

Thus in each case the elements $\varphi(g), \varphi(h)$ are conjugate in B; let $\varphi(g)^b = \varphi(h)$ with $b \in B$. Since also $\varphi(g)^{\delta} = \varphi(h)$ with $\delta \in \overline{\varphi(G')}$, we have $b \in C_{\overline{B}}(\varphi(g))\overline{\varphi(G')}$. As in the Theorem we have $\overline{\varphi(G')} = \overline{B'}\varphi(G')$, and so by Lemma 6 we have

$$b \in C_{\overline{B}}(\varphi(g))\overline{B'}\varphi(G') = C_B(\varphi(g))\overline{B'}\varphi(G').$$

It follows that

$$b \in C_B(\varphi(g))(\overline{B'} \cap B)\varphi(G') = C_B(\varphi(g))B'\varphi(G'),$$

and, since $B' \leqq \varphi(G')$ by Lemma 2, we can write $b = c_1\varphi(d)$ with $c_1 \in C_B(\varphi(g))$ and $d \in G'$. Since $\varphi(h) = \varphi(g)^{\varphi(d)} = \varphi(g^d)$ we have $h = g^d$, and the result is proved.

It remains now only to recall that every non-trivial element of G of length at most $p+1$ is conjugate in G to an element of $\Gamma \cup T$; Hypotheses (i) of our Theorem follows from this and Lemma 7. This completes our verification that each group $G \in \mathscr{X}$ satisfies the hypotheses of the Theorem, and therefore completes the proof that these groups are conjugacy separable.

References

[1] R. I. GRIGORCHUK, On the Burnside problem for periodic groups. Functional Anal. Appl. **14,** 53–54 (1980).
[2] R. I. GRIGORCHUK, The growth degrees of finitely generated groups and the theory of invariant means. Math. USSR Izv. **25,** 259–300 (1985).
[3] R. I. GRIGORCHUK, On the growth degrees of p-groups and torsion-free groups. Math. USSR-Sb. **54,** 185–205 (1986).
[4] N. GUPTA and S. SIDKI, On the Burnside problem for periodic groups. Math. Z. **182,** 385–388 (1983).
[5] J. C. C. MCKINSEY, The decision problem for some classes of sentences without quantifiers. J. Symbolic Logic **8,** 61–76 (1943).
[6] S. SIDKI, On a 2-generated infinite 3-group: subgroups and automorphisms. J. Algebra **110,** 24–55 (1987).

Eingegangen am 12. 8. 1996

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