Math. Proc. Camb. Phil. Soc. (1998), **123**, 227
Printed in the United Kingdom

Conjugacy separability of certain Bianchi groups and HNN extensions

By J. S. WILSON
School of Mathematics and Statistics, University of Birmingham

and P. A. ZALESSKII
Institute of Engineering Cybernetics, Academy of Sciences of Belarus, Minsk, Belarus

(*Received* 20 *June* 1996)

1. *Introduction*

The *Bianchi groups* are the groups $\mathrm{PSL}_2(\mathcal{O}_d)$, where $\mathcal{O}_d$ denotes the ring of integers of the field $\mathbb{Q}(\sqrt{-d})$ for each square-free positive integer d. These groups have long been of interest, not only because of their intrinsic interest as abstract groups, but also because they arise naturally in number theory and geometry. For a discussion of their algebraic properties we refer the reader to Fine [**4**]. Among the groups $\mathrm{PSL}_2(R)$, with R the ring of integers of an algebraic number field, they are distinguished by the nature of their normal subgroup structure. It was shown by Serre [**20**] that if R is not isomorphic to $\mathbb{Z}$ or $\mathcal{O}_d$, then for every normal subgroup K of $\mathrm{SL}_2(R)$ there is an ideal I of R such that the image in $\mathrm{SL}_2(R)/K$ of the kernel of the natural map from $\mathrm{SL}_2(R)$ to $\mathrm{SL}_2(R/I)$ is central and isomorphic to a subgroup of the group of roots of 1 in R. On the other hand, the group $\mathrm{PSL}_2(\mathbb{Z})$ and the Bianchi groups have many subgroups of finite index which are not of the above type: this follows easily from the fact that $\mathrm{PSL}_2(\mathbb{Z})$ is a free product of a group of order 2 and a group of order 3, and the fact, proved by Grunewald and Schwermer [**6**], that each Bianchi group has a normal subgroup of finite index which can be mapped epimorphically to a non-abelian free group.

Among the Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_d)$, the ones which have proved most amenable to study are those for which $\mathcal{O}_d$ is a Euclidean domain. These groups, the groups $\mathrm{PSL}_2(\mathcal{O}_d)$ with $d = 1, 2, 3, 7, 11$, are sometimes called the Euclidean Bianchi groups. Our object here is to give another illustration that four of these groups have many normal subgroups of finite index. A group G is said to be *conjugacy separable* if whenever a, b are non-conjugate elements of G there is some finite quotient group of G in which the images of a, b fail to be conjugate. The notion of conjugacy separability owes its importance to the fact, first pointed out by Mal'cev [**14**], that the conjugacy problem has a positive solution in finitely presented conjugacy separable groups. It is well known that $\mathrm{PSL}_2(\mathbb{Z})$ is conjugacy separable. We shall prove the following result.

Theorem 1. *The Bianchi group* $\mathrm{PSL}_2(\mathcal{O}_d)$ *is conjugacy separable for* $d = 1, 2, 7, 11$.

Since Theorem 1 holds because of the existence of normal subgroups of finite index which are not closely related to kernels of maps to groups $\mathrm{PSL}_2(\mathcal{O}_d/I)$, and which therefore have no immediate number-theoretic significance, we approach the proof with group-theoretic methods. These depend on characterizations, due to Fine [**4**], of $\mathrm{PSL}_2(\mathcal{O}_1)$ as an amalgamated free product and of $\mathrm{PSL}_2(\mathcal{O}_d)$ for $d = 2, 7, 11$ as an HNN extension. There are results asserting that, under fairly stringent conditions on the free factors G_1, G_2, an amalgamated free product $G_1 *_H G_2$ with a cyclic amalgamated subgroup H is conjugacy separable (see [**3**, **19**, **17**]). In the expression of $\mathrm{PSL}_2(\mathcal{O}_1)$ as an amalgamated free product $G_1 *_H G_2$, the amalgamated subgroup H is the natural image of $\mathrm{PSL}_2(\mathbb{Z})$, and although this is not cyclic, there is additional information available on G_1, G_2 and the embeddings of G_1, G_2, H in $\mathrm{PSL}_2(\mathcal{O}_1)$. We recall that the *profinite topology* on a group X is the topology having the family of all cosets of subgroups of finite index in X as a base of open sets; a subgroup Z is closed in this topology if and only if it equals the intersection of all subgroups of finite index containing it, and the profinite topology on X induces a (subspace) topology on a subgroup Z which is generally weaker than the profinite topology on Z. A group X is residually finite if and only if the profinite topology is Hausdorff, and X is conjugacy separable if and only if each of its conjugacy classes is closed in the profinite topology. We shall say that the profinite topology on an amalgamated free product $G = G_1 *_H G_2$ is *efficient* if G is residually finite, the profinite topology on G induces the profinite topology on G_1, G_2, H, and G_1, G_2, H are closed in the profinite topology on G. We shall show that $\mathrm{PSL}_2(\mathcal{O}_1)$ satisfies the hypotheses of the following result.

THEOREM 2 (*a*). *Let $G = G_1 *_H G_2$ be an amalgamated free product satisfying the following conditions:*

(i) *the profinite topology on G is efficient;*
(ii) *G_1, G_2, H are finitely generated virtually free groups;*
(iii) *$H \cap gHg^{-1}$ is cyclic for all $g \in G \setminus G_2$;*
(iv) *there exist a conjugacy separable group T and an epimorphism $\tau\colon G \to T$ such that $\tau|_{G_1}$ is injective and $\tau(\{g \in G \mid gHg^{-1} \leqslant G_1\}) = T$.*

Then G is conjugacy separable.

We shall also prove a corresponding result for HNN extensions and show that its hypotheses are satisfied by the groups $\mathrm{PSL}_2(\mathcal{O}_d)$ with $d = 2, 7, 11$. We say that the profinite topology on an HNN extension $G = K *_f \langle t\rangle$ is efficient if G is residually finite, the profinite topology on G induces the profinite topology on K and the associated subgroups H, H_1, and K, H, H_1 are closed in the profinite topology on G.

THEOREM 2 (*b*). *Let $G = K *_f \langle t\rangle$ be an HNN extension such that*

(i) *the profinite topology on G is efficient;*
(ii) *K and the associated subgroups H, H_1 are finitely generated virtually free groups;*
(iii) *$H \cap gHg^{-1}$ is cyclic for all $g \in G \setminus H$;*
(iv) *there exist a conjugacy separable group T and an epimorphism $\tau\colon G \to T$ such that $\tau|_K$ is injective and $\tau(\{g \in G \mid gHg^{-1} \leqslant K\}) = T$.*

Then G is conjugacy separable.

It is reasonable to conjecture that all of the Bianchi groups are conjugacy separable. However a proof would require entirely different techniques from those used here. The group $\mathrm{PSL}_2(\mathcal{O}_3)$ cannot be written either as a non-trivial amalgamated free product or as an HNN extension, and, while it is possible to write each group $\mathrm{PSL}_2(\mathcal{O}_d)$ with $d \neq 1, 2, 3, 7, 11$ as a non-trivial amalgamated free product $G_1 *_H G_2$, the structure of the groups G_i seems hard to determine except for a few small values of d.

Apart from a result of Dyer [**3**], asserting that an HNN extension of a conjugacy separable group with finite associated subgroups is conjugacy separable, and Theorem 2 (b) above, very little seems to be known about conjugacy separability of HNN extensions. A conspicuous gap in our knowledge concerns HNN extensions with cyclic associated subgroups. We shall show how the proof of Theorem 2 (b) can be modified to yield the following result, which is similar in character to results of [**19**, **17**] on amalgamated free products.

THEOREM 3. *Let $G = K *_f \langle t\rangle$ be an HNN extension with cyclic associated subgroups such that the profinite topology on G is efficient. If in addition K is either a finitely generated virtually free group or a virtually polycyclic group, then G is conjugacy separable.*

We shall prove Theorems 2 (a), 2 (b) and 3 by considering the standard trees on which amalgamated free products and HNN extensions act and the standard profinite trees on which the profinite completions of the groups act. The necessary information on abstract and profinite amalgamated free products and HNN extensions and the associated trees is given in Section 2. For a fuller account, we refer the reader to Serre [**21**] and Zalesskii and Melnikov [**23**, **24**]. In Section 2 we also collect some properties of virtually free groups which play an important part in our proofs. Two of these are new and perhaps of independent interest. Theorems 2 (a), 2 (b) and 3 are proved in Section 3. In Section 4 we study the Bianchi groups occurring in Theorem 1. One fact which emerges (in Lemma 4·2 (v) and Lemma 4·3 (v)) is that the profinite topology on each of these groups induces the profinite topology on the natural image of $\mathrm{PSL}_2(\mathbb{Z})$, so that the embedding of $\mathrm{PSL}_2(\mathbb{Z})$ in $\mathrm{PSL}_2(\mathcal{O}_d)$ induces an embedding of profinite completions, for $d = 1, 2, 7, 11$. This sheds a small amount of light on a remark of Lubotsky at the end of [**12**]. The results of Section 4 show that the Bianchi groups in Theorem 1 satisfy the hypotheses of Theorem 2 (a) and Theorem 2 (b), and so are conjugacy separable.

This paper was written while the second author was visiting the University of Birmingham. He would like to thank the School of Mathematics and Statistics for its support and warm hospitality.

2. *Preliminary results*

2·1. *Completions and standard trees.* In Section 3, we shall be concerned with amalgamated free products and HNN extensions, with their profinite completions and with the trees on which they act. We begin here by describing briefly the main definitions and the results that we shall use.

To each free amalgamated product $G = G_1 *_H G_2$ of abstract groups there corresponds a standard tree $S(G)$, constructed as follows: its vertex set is $V(S(G)) = G/G_1 \cup G/G_2$, its edge set is $E(S(G)) = G/H$, and the initial and terminal vertices of an edge $e = gH$ are respectively gG_1 and gG_2; here we write X/Y to mean the set of cosets $\{xY \mid x \in X\}$ of a subgroup Y in a group X. The group G has a natural action (on the left) on its standard tree.

Let $G = K *_f \langle t\rangle$ be an HNN extension of abstract groups: thus K is a group (called the base group of the HNN extension), $f\colon H \to H_1$ is an isomorphism between two subgroups H, H_1 of K (called the associated subgroups), and G is generated by K and t subject to the relations $tht^{-1} = f(h)$ for all $h \in H$. The standard tree $S(G)$ of G is constructed as follows: its vertex set is $V(S(G)) = G/K$, its edge set is $E(S(G)) = G/H$, and the initial and terminal vertices of an edge $e = gH$ are respectively gK and gtK. Again G has a natural action on its standard tree. For further details and general properties of trees acted on by abstract groups, we refer the reader to Serre [**21**].

Both amalgamated free products and HNN extensions may be defined in terms of universal properties. It is convenient to use the corresponding universal properties when defining profinite amalgamated free products and HNN extensions.

Let Γ_1 and Γ_2 be profinite groups with a common closed subgroup Δ. The profinite amalgamated free product of Γ_1 and Γ_2 with the amalgamated subgroup Δ consists of a profinite group Γ and two homomorphisms $i_1\colon \Gamma_1 \to \Gamma$, $i_2\colon \Gamma_2 \to \Gamma$ which agree on Δ and have the following universal property: for any profinite group Ω and any pair of homomorphisms $\phi_1\colon \Gamma_1 \to \Omega$, $\phi_2\colon \Gamma_2 \to \Omega$ such that $\phi_1|_\Delta = \phi_2|_\Delta$, there is a unique homomorphism $\phi_0\colon \Gamma \to \Omega$ such that $\phi_1 = \phi_0 i_1$ and $\phi_2 = \phi_0 i_2$. In order to verify that Γ is the profinite amalgamated free product, it is sufficient to check this universal property when Ω is finite.

Let Δ be a profinite group and let $f\colon A \to A_1$ be a continuous isomorphism between closed subgroups A, A_1 of Δ. The profinite HNN extension $\Gamma = \Delta \sqcup_f \overline{\langle t\rangle}$ of Δ with respect to f consists of a profinite group Γ, an element $t \in \Gamma$, and a homomorphism $i\colon \Delta \to \Gamma$ satisfying the following universal property: for any profinite group Ω, any $s \in \Omega$ and any homomorphism $\phi\colon \Delta \to \Omega$ satisfying $s(\phi(a))s^{-1} = \phi f(a)$ for all $a \in A$, there is a unique homomorphism $\phi_0\colon \Gamma \to \Omega$ which satisfies $\phi = \phi_0 i$ and maps t to s. Just as for profinite amalgamated products, to verify that Γ is the profinite HNN extension it is sufficient to check the universal property when Ω is finite.

Corresponding to each profinite amalgamated free product and each profinite HNN extension there is a standard profinite tree. The standard profinite tree $S(\Gamma)$ of a profinite amalgamated product $\Gamma = \Gamma_1 \sqcup_\Delta \Gamma_2$ has vertex set $V(S(\Gamma)) = \Gamma/\Gamma_1 \dot\cup \Gamma/\Gamma_2$, edge set $E(S(\Gamma)) = \Gamma/\Delta$, and the edge $e = \gamma\Delta$ has initial and terminal vertices $\gamma\Gamma_1$ and $\gamma\Gamma_2$ respectively. Similarly, the standard profinite tree $S(\Gamma)$ of a profinite HNN extension $\Gamma = \Delta \sqcup_f \overline{\langle t\rangle}$ with first associated subgroup A has vertex set $V(S(\Gamma)) = \Gamma/\Delta$, edge set $E(S(\Gamma)) = \Gamma/A$, and the initial and terminal vertices of an edge $e = gA$ are $g\Delta$ and $gt\Delta$ respectively. In both of these cases, $V(S(\Gamma))$ and $E(S(\Gamma))$ are profinite spaces (that is, they are compact Hausdorff totally disconnected topological spaces) and the natural action of Γ on $S(\Gamma)$ is continuous. For further information about standard profinite trees we refer the reader to [**23**, **24**], where the properties of profinite trees have been studied in a somewhat wider context.

Let $G = G_1 *_H G_2$ be an amalgamated free product of abstract groups and suppose that G is residually finite. It follows from the universal property of the profinite amalgamated free product that the profinite completion $\widehat{G}$ of G is equal to $\overline{G}_1 \sqcup_{\overline{H}} \overline{G}_2$, where $\overline{G}_1, \overline{H}, \overline{G}_2$ denote closures in $\widehat{G}$. In particular, if the profinite topology on G induces the profinite topologies on G_1, G_2, H, we have $\widehat{G} = \widehat{G}_1 \sqcup_{\widehat{H}} \widehat{G}_2$. We consider the standard tree $S(G)$ and the standard profinite tree $S(\widehat{G})$. It is easy to see that if G_1,G_2 and H are closed in the profinite topology on G, then $S(G)$ is naturally embedded in $S(\widehat{G})$; and in fact the image of $S(G)$ is dense in $S(\widehat{G})$.

Now suppose instead that $G = K *_f \langle t\rangle$ is an HNN extension of abstract groups which is residually finite. Let H, H_1 be the associated subgroups of G. From the universal property of the profinite HNN extension we have $\widehat{G} = G = \overline{K} \sqcup_{\bar{f}} \overline{\langle t\rangle}$, where $\overline{K}, \overline{\langle t\rangle}$ denote closures in $\widehat{G}$ and $\bar{f}$ is the isomorphism of the closures $\overline{H}, \overline{H}_1$ of H, H_1 induced by f; these closures are isomorphic, since $t\overline{H}t^{-1} = \overline{H}_1$. If the profinite topology on G induces the profinite topologies on K, H, H_1, we have $\widehat{G} = G = \widehat{H} \sqcup_{\hat{f}} \widehat{\langle t\rangle}$, where $\hat{f}$ is the isomorphism of profinite completions induced by f. We consider the standard trees $S(G)$, $S(\widehat{G})$ corresponding to $G = K *_f \langle t\rangle$ and $\widehat{G} = \overline{K} \sqcup_{\bar{f}} \overline{\langle t\rangle}$. It is easy to see that if K, H and H_1 are closed in the profinite topology on G, then $S(G)$ is naturally embedded in $S(\widehat{G})$, and again the image of $S(G)$ is dense in $S(\widehat{G})$.

We shall need the following results; we quote them in a form tailored to our purposes.

Proposition 2·1. ([**19**], Lemma 2·8). (*a*) *Let $G = G_1 *_H G_2$ be an amalgamated free product such that G is residually finite and G_1, G_2, H are closed in the profinite topology on G. If $a \in G$ and a is conjugate to an element of $G_1 \bigcup G_2$ in $\widehat{G}$, then a is conjugate to an element of $G_1 \bigcup G_2$ in G.*

(*b*) *Let $G = K *_f \langle t\rangle$ be an HNN extension such that G is residually finite and such that K and the associated subgroups are closed in the profinite topology on G. If $a \in G$ and a is conjugate to an element of K in $\widehat{G}$, then a is conjugate to an element of K in G.*

Proposition 2·2. ([**23**], Theorem 3·12). (a) *Let $\Gamma = \Gamma_1 \sqcup_\Delta \Gamma_2$ be a profinite amalgamated free product and let $\gamma \in \Gamma$. If either* (i) *$j = 1$ and $\gamma \in \Gamma \setminus \Gamma_1$ or* (ii) *$j = 2$, then $\Gamma_1 \bigcap \gamma\Gamma_j\gamma^{-1} \leqslant \mu\Delta\mu^{-1}$ for some $\mu \in \Gamma_1$.*

(*b*) *Let $\Gamma = \Delta \sqcup_f \overline{\langle t\rangle}$ be a profinite HNN extension, with first associated subgroup A. If $\gamma \in \Gamma \setminus \Delta$, then $\Delta \bigcap \gamma\Delta\gamma^{-1} \leqslant \mu A\mu^{-1}$ for some $\mu \in \Delta$.*

2·2. *Properties of virtually free groups.* Here we collect some results concerning subgroups of virtually free groups, beginning in Proposition 2·3 with some which are either known or easy extensions of known results. We recall that a group G is *subgroup separable* if whenever H is a finitely generated subgroup and g is an element of $G \setminus H$, there is a normal subgroup N of finite index in G such that $g \notin HN$; equivalently, G is subgroup separable if each finitely generated subgroup H of G is closed in the profinite topology on G. If G is residually finite, G is subgroup separable if and only if the intersection with G of the closure of H in $\widehat{G}$ equals H, for each finitely generated subgroup H.

Proposition 2·3. *Let G be a finitely generated virtually free group. Then*

(*a*) *G is conjugacy separable;*

(*b*) *G is subgroup separable;*

(*c*) *the profinite topology on G induces the profinite topology on each finitely generated subgroup of G;*

(*d*) *for each finitely generated subgroup H of G there is a free subgroup F of finite index in G such that $F = (H \bigcap F) * R$, for some subgroup R;*

(*e*) *for each pair H_1, H_2 of finitely generated subgroups of G one has $\overline{H_1H_2} \bigcap G = H_1H_2$ (where $\overline{H_1H_2}$ is the closure of H_1H_2 in $\widehat{G}$).*

Assertion (*a*) is a theorem of Dyer [**2**]. Assertion (*b*) in the case when G is free is a result of M. Hall [**7**], and the general case follows immediately. Assertion (*c*) follows from (*b*) since if H is finitely generated then so are the subgroups of finite index in H, and (*d*) follows directly from another theorem of M. Hall (see [**1**], Theorem 1 or Lyndon and Schupp [**13**], Proposition I·3·10). Assertion (*e*) in the case when G is free is a theorem of Niblo [**15**] (see also [**18**] for a more general result); and again the general case follows immediately.

Proposition 2·4. *Let G be a finitely generated virtually free group and let H_1, H_2 be finitely generated subgroups of G. Then $\overline{H_1 \cap H_2} = \overline{H_1} \cap \overline{H_2}$.*

Proof. We must prove that $\overline{H_1 \cap H_2} \geqslant \overline{H_1} \cap \overline{H_2}$, since, clearly $\overline{H_1 \cap H_2} \leqslant \overline{H_1} \cap \overline{H_2}$. Suppose first that there is a subgroup F of finite index such that $\overline{H_1 \cap H_2 \cap F} \geqslant \overline{H_1 \cap F} \cap \overline{H_2 \cap F}$. For $i = 1, 2$ the subgroup $(\overline{H_i \cap F})H_i$ is closed in $\widehat{G}$, since $H_i \cap F$ contains a subgroup of finite index in H_i, and so $\overline{H_i} = (\overline{H_i \cap F})H_i$. Let $w \in \overline{H_1} \cap \overline{H_2}$, and write $w = u_1h_1 = u_2h_2$ with $u_i \in \overline{H_i \cap F}$, $h_i \in H_i$, for $i = 1, 2$. Thus

$$h_2h_1^{-1} = u_2^{-1}u_1 \in (\overline{H_1 \cap F})(\overline{H_2 \cap F}) \cap G.$$

By Proposition 2·3 (*e*) we have $(\overline{H_1 \cap F})(\overline{H_2 \cap F}) \cap G = (H_1 \cap F)(H_2 \cap F)$. Thus we can find $v_1 \in H_1 \cap F, v_2 \in H_2 \cap F$ such that $v_2^{-1}v_1 = u_2^{-1}u_1$, and the element $k = u_2v_2^{-1} = u_1v_1^{-1}$ satisfies

$$k \in \overline{H_1 \cap F} \cap \overline{H_2 \cap F} \leqslant \overline{H_1 \cap H_2 \cap F}.$$

Since $v_1h_1 = v_2h_2 \in H_1 \cap H_2$, we conclude that $w = u_1h_1 = kv_1h_1 \in \overline{H_1 \cap H_2}$, as required.

Now we return to the general case. By Proposition 2·3 (*d*) there is a free subgroup F of finite index in G such that $F = (H_1 \cap F) * R$ for some subgroup R. Since the profinite topology on G induces the profinite topology on F, it will suffice from the above paragraph to prove the result with F replacing G and $H_i \cap F$ replacing H_i for $i = 1, 2$. In other words, we may assume that G is free and that $G = H_1 * R$ for some subgroup R. Write $H = H_1 \cap H_2$. Consider $L = G *_{H_1} G'$, where G' is a copy of G; write H_2' for the image of H_2 in G' and set $P = \langle H_2, H_2' \rangle$. By the subgroup theorem for amalgamated free products we have $P \cong H_2 *_H H_2'$. We note that all the subgroups $G, G', H_1, H_2, H_2', P, H$ of L are finitely generated (the last of these being finitely generated by Howson's theorem [**8**]; see [**13**], p. 18), and that L is a free group. It follows from Proposition 2·3 (*b*), (*c*) that each of these subgroups is closed in L and that its closure in $\widehat{L}$ is isomorphic to its profinite completion. Thus the natural maps $\alpha: \widehat{G} \amalg_{\widehat{H_1}} \widehat{G'} \to \widehat{L}$ and $\beta: \widehat{H_2} \amalg_{\widehat{H}} \widehat{H_2'} \to \overline{P}$ induced by inclusion maps from G, G', H_2, H_2' are isomorphisms. In particular, since β is an isomorphism we have $\overline{H_2} \cap \overline{H_2'} = \overline{H}$. Since α is an isomorphism we can define a homomorphism $\phi: \widehat{L} \to \overline{G}$ which maps each element of G to itself and each element of G' to its preimage under the isomorphism from G to G'. Clearly $\phi(\overline{H_1} \cap \overline{H_2}) = \phi(\overline{H_1} \cap \overline{H_2'})$, and since the restriction of ϕ to $\overline{G}$ is the identity map we conclude that $\overline{H_1} \cap \overline{H_2} = \overline{H_1} \cap \overline{H_2'}$. Thus $\overline{H_1} \cap \overline{H_2} \leqslant \overline{H_2} \cap \overline{H_2'} = \overline{H}$, as required.

A subgroup H of a group G is said to be *conjugacy distinguished* if whenever a is an element of G having no conjugate in H, there exists a normal subgroup N of finite index in G such that no conjugate of a lies in HN. Thus if G is residually finite, then H is conjugacy distinguished in G if and only if the following condition holds: whenever $a \in G$ and there is an element $\gamma \in \widehat{G}$ with $\gamma a \gamma^{-1} \in \overline{H}$ then there is an element $g \in G$ with $gag^{-1} \in H$.

PROPOSITION 2·5. *Every finitely generated subgroup H of a finitely generated virtually free group G is conjugacy distinguished.*

Proof. Let $a \in G$ and suppose that $\gamma a \gamma^{-1} \in \overline{H}$ for some $\gamma \in \widehat{G}$.

First suppose that a has finite order. Since H is virtually free, it is the fundamental group of a finite graph of finite groups by a theorem of Karrass, Pietrowski and Solitar [**10**], and its profinite completion $\widehat{H}$ is the profinite fundamental group of the same finite graph of groups (see Zalesskii and Mel'nikov [**23**], paragraph 3·3). It follows from Theorem 3·10 in [**23**] that every conjugacy class of elements of finite order of $\widehat{H}$ contains an element of H. Since the closure of H in $\widehat{G}$ is isomorphic to $\widehat{H}$ by Proposition 2·3 (c), we conclude that the conjugacy class in $\overline{H}$ of $\gamma a \gamma^{-1}$ contains an element a_1 which belongs to H. Since G is conjugacy separable by Proposition 2·3 (a) there is an element $g \in G$ with $gag^{-1} = a_1$, and the result follows.

Now suppose that a has infinite order. By Proposition 2·3 (d), there exists a free subgroup F of finite index in G such that $F = H_1 * R$ for some subgroup R, where $H_1 = H \cap F$. Since F has finite index in G we have $\widehat{G} = G\overline{F}$, and so, replacing a by a conjugate in G, we can assume that $\gamma \in \overline{F}$. Pick $n \in \mathbb{N}$ such that $a^n \in F$. It follows from Proposition 2·1 that a^n is conjugate in F to an element of H_1 or R; since $\overline{F} = \widehat{F} = \widehat{H_1} \sqcup \hat{R}$, it follows from Proposition 2·2 that no non-trivial element of $\hat{R}$ can be conjugate to an element of $\widehat{H_1}$ in $\widehat{F}$. Thus there is an element $g \in F$ with $ga^n g^{-1} \in H_1$. Write $a_1 = gag^{-1}$ and $\gamma_1 = \gamma g^{-1}$. We have $a_1^n \in H_1$ and $\gamma_1 a_1^n \gamma_1^{-1} \in \overline{H}_1$, and so $\overline{H}_1 \cap \gamma_1 \overline{H}_1 \gamma_1^{-1}$ is non-trivial. It follows from Proposition 2·2 that $\gamma_1 \in \overline{H}_1$, and since $\gamma_1 a_1 \gamma_1^{-1} = \gamma a \gamma^{-1} \in \overline{H}$ we have $a_1 = gag^{-1} \in \overline{H} \cap G = H$, as required.

3. *Proof of Theorems* 2 (a), 2 (b) *and Theorem 3*

We shall prove Theorems 2 (a), 2 (b) simultaneously and afterwards explain the modifications necessary for the proof of a result which implies Theorem 3. To simplify the exposition, in Theorem 2 (b) we define G_1, G_2 and H to be respectively K, the trivial subgroup, and the first associated subgroup. Thus G is either (a) an amalgamated free product $G_1 *_H G_2$ or (b) an HNN extension $G_1 *_f \langle t \rangle$ with first associated subgroup H, and our hypotheses are as follows:

(i) the profinite topology on G is efficient;
(ii) G_1, G_2, H are finitely generated and virtually free;
(iii) $H \cap gHg^{-1}$ is cyclic for all $g \in G \setminus G_2$ if G is an amalgamated free product and for all $g \in G \setminus H$ if G is an HNN extension;
(iv) there exist a conjugacy separable group T and an epimorphism $\tau \colon G \to T$ such that $\tau|_{G_1}$ is injective and $\tau(\{g \in G \mid gHg^{-1} \leqslant G_1\}) = T$.

Let $a, b \in G$, and assume that $\gamma a \gamma^{-1} = b$ for some $\gamma \in \widehat{G}$. Our aim is to show that $gag^{-1} = b$ for some $g \in G$. Our strategy is to replace a, b repeatedly by conjugates under G until we can make effective use of hypothesis (iii) or hypothesis (iv).

Case 1. One of the elements a, b is conjugate to an element of G_1 or G_2.

In this case, by Proposition 2·1, each of a, b is conjugate to an element of G_1 or G_2, and so we may assume that $a, b \in G_1 \cup G_2$. If γ belongs to the closure in $\widehat{G}$ of the one of these subgroups which contains a, then the result follows from hypothesis (ii) and the conjugacy separability of G_1, G_2. Otherwise, by Proposition 2·2, we have $a \in \alpha\overline{H}\alpha^{-1}, b \in \beta\overline{H}\beta^{-1}$ for some $\alpha, \beta \in \overline{G}_1 \cup \overline{G}_2$. Then a and b are conjugate to elements of H by Proposition 2·5, and so we may assume that $a, b \in H$. We can now use hypothesis (iv). The elements a, b are conjugate in $\widehat{G}$, and so, applying the epimorphism from $\widehat{G}$ to $\widehat{T}$ induced by τ, we see that $\tau(a), \tau(b)$ are conjugate in $\widehat{T}$. Since T is conjugacy separable, it follows that $\tau(a), \tau(b)$ are conjugate in T, and that there is an element $z \in \{g \in G \mid gHg^{-1} \leqslant G_1\}$ such that zaz^{-1}, b have the same image under τ. However $zaz^{-1}, b \in G_1$, and since $\tau|_{G_1}$ is injective we must have $zaz^{-1} = b$. This concludes the treatment of Case 1.

Case 2. Neither of a, b is conjugate to an element of G_1 or G_2.

In this case we shall study the actions of $G, \widehat{G}$ on the associated trees $S(G), S(\widehat{G})$.

Consider the standard tree $S(G)$ corresponding to the amalgamated free product $G = G_1 *_H G_2$ (resp. the HNN extension $G = G_1 *_f \langle t\rangle$) and the standard profinite tree $S(\widehat{G})$ corresponding to the profinite amalgamated product $\widehat{G} = \widehat{G}_1 \amalg_{\widehat{H}} \widehat{G}_2$ (resp. the profinite HNN extension $\widehat{G} = \widehat{G_1} \amalg_{\hat{f}} \overline{\langle t\rangle}$). From (i), the natural map from $S(G)$ to $S(\widehat{G})$ is an embedding and we shall regard this as inclusion. The hypothesis of Case 2 implies that a, b act freely on $S(G)$. Thus we have $m_a, m_b > 0$, where

$$m_a = \min\,\{l(v, av) \mid v \in V(S(G))\}, \qquad m_b = \min\,\{l(v, bv) \mid v \in V(S(G))\},$$

and where $l(u, v)$ denotes the distance between two vertices u, v in $S(G)$. Write

$$V_a = \{v \in V(S(G)) \mid l(v, av) = m_a\} \quad \text{and} \quad V_b = \{v \in V(S(G)) \mid l(v, bv) = m_b\}.$$

By a theorem of Tits (cf. [**21**], proposition 24), there are doubly infinite paths T_a, T_b in $S(G)$ having vertex sets V_a, V_b respectively, and moreover a, b act freely on T_a, T_b as translations of lengths m_a, m_b, respectively. Let T_1 and T_2 be finite paths in T_a and T_b of lengths m_a and m_b, respectively. Then $T_a = \langle a\rangle T_1$ and $T_b = \langle b\rangle T_2$.

Write e for the edge in $S(G)$ whose stabilizer in G is H (so that e is, in fact, H regarded as a coset of H in G). First we claim that one may assume that $e \in T_1$. To see this, consider $g_1 \in G$ such that $e \in g_1T_1$ and set $a' = g_1ag_1^{-1}$. Then a' is not conjugate to an element of G_1 or G_2, and there is a straight line $T_{a'} = g_1T_a$ corresponding to a'. Define $T_1' = g_1T_1$. Then clearly $T_{a'} = \langle a'\rangle T_1'$. Since a and b are conjugate if and only if a' and b are conjugate, the claim follows.

Consider the profinite subgraphs of $S(\widehat{G})$ defined by $\overline{T}_a = \overline{\langle a\rangle}T_1$ and $\overline{T}_b = \overline{\langle b\rangle}T_2$. By proposition 2·9 in [**19**], the subgroups $\overline{\langle a\rangle}$ and $\overline{\langle b\rangle}$ act freely on $\overline{T}_a$ and $\overline{T}_b$, respectively. Since $\gamma a\gamma^{-1} = b$, the element b also acts freely on $\gamma\overline{T}_a$. By lemma 2·2 (ii) in [**19**], we have $\gamma\overline{T}_a = \overline{T}_b$, and so $\gamma e \in \overline{T}_b$. Choose $b' \in \overline{\langle b\rangle}$ such that $b'\gamma e \in T_2$. Then $b'\gamma e = ge$ for some $g \in G$, and hence $b'\gamma = g\delta$ for some $\delta \in \overline{H}$. Now $a = \gamma^{-1}b'^{-1}bb'\gamma = \delta^{-1}g^{-1}bg\delta$. Therefore, replacing b by $g^{-1}bg$ and γ by δ we can assume that γ is in $\overline{H}$.

We need to arrange that γ fixes longer paths in T_a than the path whose only edge is e. Suppose that P is a finite path in T_a which has e as one of its edges and such that $\gamma \in \overline{L}$, where L is the intersection of the stabilizers in G of the edges of P:

we shall show that γ can be replaced by an element which lies in the closure of the intersection of the edge stabilizers of a path strictly containing P.

Let e_1 be an edge of $T_a \setminus P$ connected to P, write v for the common vertex of e_1 and P, and write P_+ for the path with edges those of P together with e_1. Let $e_2 = \gamma e_1 \in \overline{T}_b$. First we note that $e_2 \in T_b$. Indeed, let e' be an edge in T_b. There is a path in $S(G)$ connecting e' to e_1, and so since e_1, e_2 share a vertex there is a path connecting e' to e_2. However if f_1, f_2 are edges of a profinite tree then there is a unique smallest profinite subtree containing f_1, f_2, from [**23**], paragraph 1.19, and so since $e_2 = \gamma e_1 \in \gamma\overline{T_a} = \overline{T_b}$ and $e' \in \overline{T_b}$, it follows that the shortest path connecting e' and e_2 lies in $\overline{T_b}$. The connected component of $\overline{T}_b$ containing e' is precisely T_b (by [**17**], lemma 4·3 (iii)), and so we conclude that $e_2 \in T_b$. Now since v is a common vertex of e_1 and e_2, we have $ge_1 = e_2$ for some g in the stabilizer G_v of v in G. If x is a vertex or edge in $S(G)$ then its stabilizer G_x in G is conjugate to G_1, G_2 or H, and so G_x is finitely generated and $\overline{G}_x$ is the stabilizer of x in $\widehat{G}$. Thus since $e_1 = g^{-1}e_2 = \gamma^{-1}e_2$ the element $\gamma_1 = \gamma g^{-1}$ is in $\overline{G}_{e_2}$. Moreover both L and G_{e_2} are finitely generated (the former by Howson's theorem [**8**]), and since they are both subgroups of the virtually free group G_v we have $\overline{G_{e_2}L} \cap G_v = G_{e_2}L$ from Proposition 2·3 (e). Therefore because $g = \gamma_1^{-1}\gamma$ we can find $h_1 \in L, h_2 \in G_{e_2}$ with $g = h_2h_1$. Set $\gamma_+ = h_1^{-1}\gamma$. Thus

$$\gamma_+e_1 = h_1^{-1}e_2 = g^{-1}h_2e_2 = g^{-1}e_2 = e_1,$$

and so $\gamma_+ \in \overline{G}_{e_1}$. We also have $\gamma_+ \in \overline{L}$. Both L and G_{e_1} are finitely generated subgroups of the virtually free group G_v and it follows from Proposition 2·4 that γ_+ is in the closure of the intersection of the edge stabilizers of the path P_+. We may therefore replace γ by γ_+ and b by $h_1^{-1}bh_1$ and so assume that γ is in the closure of the intersection of the edge stabilizers of P_+.

Let f be an edge in T_a having a vertex in common with e; in the case when G is an amalgamated free product, we choose f so that this common vertex is the coset G_1. From above, we can assume that there is a finite path P whose edges include e, f, ae, af such that $\gamma \in \overline{L}$, where L is the intersection of the stabilizers of the edges of P. Write $D = H \cap G_f$; thus $L \leqslant D \cap aDa^{-1}$ and D is a cyclic group by (iii).

Our next claim is that a normalizes L. Let N be a normal subgroup of finite index in G and consider the quotient group G/N. The subgroup LN/N has the same index, m, say, in both DN/N, $(aDa^{-1})N/N$, since these subgroups are conjugate. Thus if dN generates DN/N, then $(dN)^m$ and $(aN)(dN)^m(aN)^{-1}$ both generate LN/N, and we conclude that LN is normalized by a. However, since L is closed in H by Proposition 2·3 (b), and hence closed in G, the subgroup L is the intersection of all such subgroups LN and so it is normalized by a.

Let h be a generator of L and write $E = \langle h, a\rangle$. If h either has finite order or centralizes a then clearly the conjugacy class of a in E is finite and hence closed in G. If h has infinite order and does not centralize a then we have $aha^{-1} = h^{-1}$; in this case $\langle h^2\rangle$ is closed in H by Proposition 2.3 (b) and hence closed in G, so that the conjugacy class

$$\{kak^{-1} \mid k \in \langle h\rangle\} = \{kak^{-1}a^{-1} \mid k \in \langle h\rangle\}a = \langle h^2\rangle a$$

of a in E is closed in G.

Now $a \in E$ and $\gamma \in \overline{L} \leqslant \overline{E}$, so that $b = \gamma a \gamma^{-1} \in \overline{E}$. Moreover a, b are congruent modulo the closed normal subgroup of $\overline{E}$ generated by γ. It follows that

$$ab^{-1} \in \overline{L} \cap G = \overline{L} \cap (\overline{H} \cap G) = \overline{L} \cap H,$$

and so $ab^{-1} \in L$ by Proposition 2·3 (*b*). Therefore a, b are elements of E conjugate in $\overline{E}$, and since the conjugacy classes of E are closed it follows that a, b are conjugate in E. This completes the proof of Theorems 2 (*a*), 2 (*b*).

Now we turn to Theorem 3. We shall show that the following somewhat stronger theorem holds:

Theorem 3′

Let $G = K *_f \langle t \rangle$ be an HNN extension with cyclic associated subgroups H, H_1 such that the profinite topology on G is efficient. Suppose that K satisfies the following conditions:

(i) *K is conjugacy separable;*

(ii) *for any pair A, B of cyclic subgroups of K, the set AB is closed in K;*

(iii) *for any pair A, B of cyclic subgroups of K, the subgroups $\overline{A \cap B}$ and $\overline{A} \cap \overline{B}$ are equal;*

(iv) *every cyclic subgroup of K is conjugacy distinguished.*

Then G is conjugacy separable.

To see that Theorem 3 follows from this we need to explain why finitely generated virtually free groups and virtually polycyclic groups have the properties required of K above. The required properties of virtually free groups were given in Section 2. The conjugacy separability of virtually polycyclic groups was established by Remeslennikov [**16**] and Formanek [**5**], and property (ii) for virtually polycyclic groups K follows from a result of Lennox and Wilson [**11**]. Properties (iii) and (iv) are shown to hold for virtually polycyclic groups in [**17**].

Proof of Theorem 3′.

Let H be the first associated subgroup of G. If H is finite, the result holds from the theorem of Dyer [**3**], and so we shall assume that H is infinite. Let a, b be elements of G which are conjugate in $\widehat{G}$. The proof that a, b are conjugate in G divides into two cases, Case 1 and Case 2, just as in the proof of Theorem 2 (*b*). In Case 2, we assume that neither a nor b is conjugate to an element of K. The proof proceeds exactly as for Theorem 2 (*b*); the two references to Proposition 2·3 are replaced by references to hypothesis (ii), and hypothesis (iii) is used instead of Proposition 2·4. In Case 1, a, b are conjugate to elements of K, and the first part of the argument given for this case in Theorem 2 (*b*) shows that a, b may be assumed to lie in H.

Write $A = \langle a \rangle$ and $B = \langle b \rangle$. If N is a normal subgroup of finite index in G then AN/N and BN/N are subgroups of equal order in the cyclic group HN/N, and therefore $AN = BN$. Since A, B are closed in G it follows that $A = B$ and that $b = a$ or $b = a^{-1}$. Suppose therefore that $b = a^{-1}$. Thus if γ is an element of $\widehat{G}$ such that $\gamma a \gamma^{-1} = a^{-1}$, then $\gamma \in N_{\widehat{G}}(\overline{A})$. If $N_{\widehat{G}}(\overline{A}) = N_{\overline{K}}(\overline{A})$, then the conjugacy separability of K implies that a, a^{-1} are conjugate in K. Assume then that $N_{\widehat{G}}(\overline{A}) \neq N_{\overline{K}}(\overline{A})$. It follows from Proposition 2·5 in [**19**] that $N_{\widehat{G}}(\overline{A}) = N_{\overline{K}}(\overline{A}) \amalg_{\bar{f}} \overline{\langle t \rangle}$, where $\bar{f}$ is the isomorphism of closures induced by f; in particular, $N_{\widehat{G}}(\overline{A})$ is generated as a profinite group by $N_{\overline{K}}(\overline{A})$ and t. Since the result is clear if $tat^{-1} = a^{-1}$ we may assume that

t centralizes a. However this implies that $N_{\hat{G}}(\overline{A}) = N_{\overline{K}}(\overline{A})C_{\hat{G}}(\overline{A})$, so that there is an element $\gamma_1 \in N_{\overline{K}}(\overline{A})$ with $\gamma_1 a \gamma_1^{-1} = a^{-1}$. Because K is conjugacy separable we conclude that $sas^{-1} = a^{-1}$ for some $s \in K$, and the proof of Theorem 3′ is complete.

4. *Proof of Theorem 1*

In this section it remains to show that the hypotheses of Theorem 2 (*a*) and Theorem 2 (*b*) are satisfied by the Bianchi groups $\mathrm{PSL}_2(\mathscr{O}_d)$ with $d = 1, 2, 7, 11$. The information we require is contained in the following three lemmas.

Lemma 4·1. *Let m be a square-free integer with $m \neq 0, 1$, let $u^2 = m$, and let R be the ring of algebraic integers of $\mathbb{Q}(u)$. Let $\Gamma = \mathrm{SL}_2(R)$ and let M be the image of $\mathrm{PSL}_2(\mathbb{Z})$ in Γ. Then*

(*a*) *M is closed in the profinite topology on Γ;*

(*b*) *$N_\Gamma(M)/M$ has order 2 if $u^2 = -1$ and is trivial otherwise;*

(*c*) *if $g \in \Gamma \setminus N_\Gamma(M)$ then $M \cap M^g$ is cyclic.*

Proof. (*a*) The centre of $\mathrm{SL}_2(R)$ consists of the two matrices ± 1 and coincides with the centre of $\mathrm{SL}_2(\mathbb{Z})$. Therefore it is sufficient to show that $\mathrm{SL}_2(\mathbb{Z})$ is closed in $\mathrm{SL}_2(R)$. Let R be generated as a ring by θ, and for each integer $n > 0$ let R_n be the subring generated by $n\theta$. The group $\mathrm{SL}_2(R_n)$ has finite index in $\mathrm{SL}_2(R)$ since it contains the kernel of the natural map from $\mathrm{SL}_2(R)$ to $\mathrm{SL}_2(R/nR)$. Clearly $\mathrm{SL}_2(\mathbb{Z}) = \bigcap \mathrm{SL}_2(R_n)$, and (*a*) follows.

(*b*), (*c*) Since M is a free product of a group of order 2 and a group of order 3, the subgroup theorem for free products implies that the centralizer of each non-trivial element of M is cyclic, and, in particular, that each abelian subgroup of M is cyclic. Since the kernel of the map from $\mathrm{SL}_2(\mathbb{Z})$ to M has order 2, torsion-free abelian subgroups of $\mathrm{SL}_2(\mathbb{Z})$ are also cyclic. We note that if l is a 2×2 matrix over $\mathbb{Q}$ such that $l^2 = 0$ and if there exists a matrix $k \in \mathrm{SL}_2(\mathbb{Q})$ such that $lk = -kl$ then $l = 0$. For otherwise, conjugating l, k by a suitable element of $\mathrm{GL}_2(\mathbb{Q})$, we may assume that

$$l = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Write

$$k = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Multiplying and equating coefficients, we find that $c = 0$ and $a = -d$. But then $1 = \det(k) = -a^2$, and we have a contradiction.

Write $G = \mathrm{SL}_2(R)$, let H be the preimage of M in G, and let A be the ring of 2×2 matrices over $\mathbb{Q}$. Choose $g \in G$, and set $S = A \cap gAg^{-1}$; this is a $\mathbb{Q}$-algebra of dimension at most 4 containing $H \cap gHg^{-1}$. First suppose that $\dim S \leqslant 3$. If S is semisimple, it must be a direct sum of fields, by the Wedderburn–Artin Theorem (see [**9**], p. 41) so that S is commutative and $H \cap gHg^{-1}$ is abelian. It follows that the image of $H \cap gHg^{-1}$ in Γ is abelian, and hence cyclic. If S is not semisimple, it has an ideal $I \neq 0$ with $I^2 = 0$. Then $(1 + I) \cap H$ is a non-trivial free abelian group normalized by $H \cap gHg^{-1}$, and so is cyclic; let $h = 1 + l$ be a generator. The centralizer of h in $H \cap gHg^{-1}$ has index at most 2. However if $k \in H \cap gHg^{-1}$ and k does not centralize h then we have

$$khk^{-1} = k(1 + l)k^{-1} = (1 + l)^{-1} = 1 - l,$$

and hence $lk = -kl$, and we have a contradiction from the above paragraph. Therefore $H \cap gHg^{-1}$ centralizes h, and so its image in M centralizes the image of h. We conclude that the image of $H \cap gHg^{-1}$ is cyclic.

If $\dim(S) = 4$ then $A = gAg^{-1}$, and so to establish (*c*) it is now sufficient to prove that if $A = gAg^{-1}$ then g normalizes H. We shall do this and prove (*b*) simultaneously. Let g be an element of G which either satisfies the condition $A = gAg^{-1}$ or normalizes H, and write

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The following matrices are in A:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} -ac & a^2 \\ -c^2 & ac \end{pmatrix},$$

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} cd & d^2 \\ -c^2 & cd \end{pmatrix},$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} bd & -b^2 \\ d^2 & -bd \end{pmatrix}.$$

It follows that $a^2, b^2, c^2, d^2, ac, cd, bd \in \mathbb{Q}$. If $a \in \mathbb{Q}$ we conclude that $a, b, c, d \in \mathbb{Q} \cap R = \mathbb{Z}$ so that $g \in H$. If $a \notin \mathbb{Q}$ then since $a^2 \in \mathbb{Q}$ we must have $a = a'u$ with $a' \in \mathbb{Q}$, and hence all of $a' = au^{-1}, b' = bu^{-1}, c' = cu^{-1}, d' = du^{-1}$ are in $\mathbb{Q}$. Since $a^2 = a'^2u^2 \in \mathbb{Q} \cap R = \mathbb{Z}$, and since u^2 is square-free in $\mathbb{Z}$, a' has denominator 1. Arguing similarly for b', c', d' we see that the entries of the matrix

$$g' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

are in $\mathbb{Z}$. Therefore $1 = \det g = \det ug' = u^2 \det g'$. Thus g can only fail to be in H when $u^2 = -1$, and in this case we have $g = ig'$ with $g' \in \mathrm{GL}_2(\mathbb{Z})$, so that $g \in N_G(H)$. The quotient of two such matrices $g_1, g_2 \in N_G(H) \setminus H$ clearly lies in H, so that if $u^2 = -1$ then both $N_G(H)/H$ and $N_\Gamma(M)/M$ have order 2. This completes the proof of the lemma.

We now restrict attention to the Bianchi groups occurring in Theorem 1. We begin by studying the group $\Gamma_1 = \mathrm{PSL}_2(\mathcal{O}_1)$.

Lemma 4·2. *There exist subgroups G_1, G_2 of Γ_1, containing M, with the following properties:*

(i) *Γ_1 is isomorphic to the amalgamated free product $G_1 *_M G_2$ and $|G_2 : M| = 2$;*

(ii) *G_1, G_2 are virtually free;*

(iii) *there exist an involution $u \in G_2 \setminus M$ and an automorphism ϕ of order 2 of G_1 such that $\phi(x) = uxu^{-1}$ for all $x \in M$;*

(iv) *there is a surjective homomorphism τ from Γ_1 to the semidirect product of G_1 by $\langle\phi\rangle$ such that $\tau|_{G_1}$ is the identity map;*

(v) *G_1, G_2, M are closed in Γ_1 with respect to the profinite topology, and the profinite topology on Γ_1 induces the profinite topology on each of the subgroups G_1, G_2, M.*

Proof. A description of Γ_1 as an amalgamated free product is given in Fine [**4**], pp. 83–85. It is shown that there are generators s, a, u, v of Γ_1 such that the group

G_1 generated by s, a, v has the presentation

$$\langle s, a, v \mid v^3 = s^3 = a^2 = (vs)^2 = (va)^2 = 1\rangle,$$

such that the group G_2 generated by s, a, u has the presentation

$$\langle s, a, u \mid u^2 = s^3 = a^2 = (ua)^2 = (us)^2 = 1\rangle,$$

and such that $G_1 \cap G_2 = M = \langle s, a\rangle$ and the natural map from $G_1 *_M G_2$ to Γ_1 is an isomorphism. It follows from the presentation of G_2 that u normalizes M, so that $|G_2 : M| = 2$ (and hence $G_2 = N_{\Gamma_1}(M)$ by Lemma 4.1) and G_2 is virtually free. The subgroup A of G_1 generated by v, s has presentation

$$\langle v, s \mid v^3 = s^3 = (vs)^2 = 1\rangle$$

and is isomorphic to the alternating group A_4, and the subgroup B generated by v, a has presentation

$$\langle v, a \mid v^3 = a^2 = (va)^2 = 1\rangle$$

and is isomorphic to the symmetric group Σ_3; and the natural map from $A *_{\langle v\rangle} B$ to G_1 is an isomorphism. Therefore, being isomorphic to an amalgamated free product with finite free factors, G_1 is virtually free (see [**21**], proposition 11 on p. 120). Now if P_1, P_2 are distinct Sylow 3-subgroups of Σ_4, there is a transposition in Σ_4 which acts as inversion on both P_1 and P_2. Hence there is an automorphism of A which maps v, s to their inverses, and this extends to an automorphism ϕ of G_1 fixing a. Clearly ϕ has order 2 and $\phi(x) = uxu^{-1}$ for all $x \in M$. Thus (iii) holds. The map ψ defined by $xu^\epsilon \mapsto x\phi^\epsilon$ for $x \in M, \epsilon = 0, 1$ is an isomorphism from G_2 to $M\langle\phi\rangle$ which fixes M pointwise, and thus there is a homomorphism τ from Γ_1 which agrees with ψ on G_2 and with the identity map on G_1.

Let N be a subgroup of G_1 of finite index, and let N_1 be the preimage of N under the map τ. Thus N_1 is a subgroup of Γ_1 of finite index, and we have $N_1 \cap G_1 = N$ since $\tau_{G_1} = \mathrm{id}$. It follows that the profinite topology on Γ_1 induces the profinite topology on G_1, and a similar argument shows that it induces the profinite topology on G_2. Since $|G_2\colon M| = 2$, it also follows that the profinite topology is induced in M. The fact that M is closed was proved in Lemma 4·1, and it follows that G_2 is closed. Finally, let L be the preimage of G_1 under τ; thus $|\Gamma_1\colon L| = 2$. Consider the homomorphism $\tau_1\colon L \to L$ defined by $x \mapsto \tau(x)$. The set of elements $x \in L$ whose images under τ_1 and the identity map on L coincide is G_1, and since these two maps are continuous it follows that G_1 is closed.

LEMMA 4·3. *Let $d = 2, 7$ or 11 and write $\Gamma_d = \mathrm{PSL}_2(\mathcal{O}_d)$. There exist a subgroup K_d of Γ_d, containing M, and an element $u \in \Gamma_d$, with the following properties:*

(i) *Γ_d is isomorphic to an HNN extension $K_d *_f \langle u\rangle$ with first associated subgroup M;*

(ii) *K_d is virtually free;*

(iii) *there is an automorphism ψ of K_d of order 2 such that $\psi(x) = uxu^{-1}$ for all $x \in M$;*

(iv) *there is a surjective homomorphism τ from Γ_d to the semidirect product of K_d by $\langle\psi\rangle$ such that $\tau|_{K_d}$ is the identity map;*

(v) *K_d is closed in Γ_d with respect to the profinite topology on Γ_d, and the profinite topology on Γ_d induces the profinite topology on K_d and M.*

Proof. We begin with assertions (i) and (iii), and treat the three cases $d = 2, 7, 11$ separately. Assertion (i) may be found in Fine [**4**], pp. 87–89; a complete proof was given for the case $d = 2$ and the details for $d = 7, 11$ were left for the reader. Assertion (iii) for $d = 2$ was given in Zalesskii [**22**]. We shall need to fill in some of the details in Fine's arguments since they are required for the proof of (iii). In all cases, K_d will be an amalgamated free product of finite groups and therefore it is virtually free.

First suppose that $d = 2$. Then Γ_2 has a presentation

$$\Gamma_2 = \langle a, h, z, v \mid a^2 = (ah)^3 = v^2 = (av)^2 = (hv)^3 = 1, z^{-1}hz = h, z^{-1}az = v\rangle$$

such that a, h are generators of M (see Fine [**4**], p. 88). Let K_2 be the subgroup generated by a, h, v; thus K_2 has the presentation

$$K_2 = \langle a, h, v \mid a^2 = v^2 = (av)^2 = (ah)^3 = (hv)^3 = 1\rangle.$$

Set $u = z^{-1}$ and write f for the isomorphism with domain M defined by $a \mapsto v, h \mapsto h$. Thus $f(x) = uxu^{-1}$ for all $x \in M$, and the homomorphism from $K_2 *_f \langle t\rangle$ to Γ_2 which is the identity on K_2 and maps t to u is an isomorphism. It is shown in [**4**] (loc. cit.) that K_2 is a free product of the alternating group A_4 and a non-cyclic group of order 4, with a subgroup of order 2 amalgamated. From the symmetry of the above presentation there is an automorphism ϕ of K_2 which fixes h and exchanges a, v; and this clearly has order 2 and satisfies $\phi(x) = f(x)$ for all $x \in M$.

Next suppose that $d = 7$. Then Γ_7 has a presentation

$$\Gamma_7 = \langle a, v, s, m, w \mid a^2 = v^3 = (av)^2 = 1, av = ms, w^{-1}aw = m, w^{-1}sw = v\rangle$$

such that a, s are generators of M (see Fine [**4**], p. 86). Let K_7 be the subgroup generated by a, v, s, m; thus K_7 has the presentation

$$K_7 = \langle a, v, s, m \mid a^2 = v^3 = (av)^2 = 1, av = ms, m^2 = s^3 = 1\rangle.$$

Set $u = w^{-1}$ and write f for the isomorphism with domain M defined by $a \mapsto m, s \mapsto v$. Thus $f(x) = uxu^{-1}$ for all $x \in M$, and the homomorphism from $K_7 *_f \langle t\rangle$ to Γ_7 which is the identity on K_7 and maps t to u is an isomorphism. From the above presentation of K_7 it follows that K_7 is the amalgamated free product of the subgroups $\langle a, v \mid a^2 = v^3 = (av)^2 = 1\rangle$ and $\langle m, s \mid m^2 = s^3 = (ms)^2 = 1\rangle$, both of which are isomorphic to Σ_3, with the subgroups $\langle av\rangle$ and $\langle ms\rangle$ amalgamated. It is clear that the map $a \mapsto m, m \mapsto a, v \mapsto s, s \mapsto v$ extends to an automorphism ϕ of K_7; evidently ϕ has order 2 and $\phi(x) = f(x)$ for all $x \in M$.

Finally, suppose that $d = 11$. Then Γ_{11} has the presentation

$$\Gamma_{11} = \langle a, t, z \mid a^2 = (at)^3 = (z^{-1}azat)^3 = [t, z] = 1\rangle,$$

where the elements a, t generate M (see Fine [**4**], p. 87). Letting $s = at, w = az, m = w^{-1}aw, v = w^{-1}sw$ and applying Tietze transformations we obtain the presentation

$$\Gamma_{11} = \langle a, s, v, m, w \mid a^2 = s^3 = (sm)^3 = 1, av = sm, m = w^{-1}aw, v = w^{-1}sw\rangle.$$

Let K_{11} be the subgroup generated by a, s, v, m; thus

$$K_{11} = \langle a, s, v, m \mid a^2 = s^3 = m^2 = v^3 = (sm)^3 = 1, av = sm\rangle,$$

so that K_{11} is the amalgamated free product of the two groups

$$\langle a, v \mid a^2 = v^3 = (av)^3 = 1\rangle,\ \langle m, s \mid m^2 = s^3 = (ms)^3 = 1\rangle,$$

each isomorphic to the alternating group A_4, with a subgroup of order 3 amalgamated. Let ϕ be the map $s \mapsto v, a \mapsto vmv^{-1}, m \mapsto s^{-1}as, v \mapsto s$. It is easy to check that $\phi(a)\phi(v) = \phi(s)\phi(m)$, and hence that ϕ extends to an automorphism of K_{11}. This automorphism has order 2, and if we define $f : M \to \phi(M)$ by $f(x) = \phi(x)$ and set $u = vw^{-1}$ then we have $f(s) = usu^{-1}, f(a) = uau^{-1}$, and hence $f(x) = uxu^{-1}$ for all $x \in M$. It is clear that the map from $K_{11} *_f \langle t\rangle$ to Γ_{11} which is the identity on K_{11} and maps t to u is an isomorphism.

The proof of (iv) and (v) now follows easily. Because the automorphism ϕ of K_d extends the map f from M, there is a natural map from $K_d *_f \langle t\rangle$ to $K_d *_\phi \langle t\rangle$, and we let τ be the composition of this map with the map from the latter group to the split extension of K_d by $\langle\phi\rangle$. Let N be a subgroup of finite index in K_d, and let N_1 be the preimage in Γ_d of N under the map τ; thus N_1 is a subgroup of finite index in Γ_d and we have $N_1 \cap K_d = N$ since $\tau|_{K_d} = \mathrm{id}$. It follows that the profinite topology on Γ_d induces the profinite topology on K_d; and since K_d is virtually free, the profinite topology on K_d induces the profinite topology on M by Proposition 2·3 (*c*). Let L be the inverse image of K_d under the map τ. Since $|\Gamma_d\colon L| = 2$ and since K_d is the set of elements $x \in L$ such that $\tau(x) = x$, it follows that K_d is closed in Γ_d. This concludes the proof of Lemma 4·3.

Proof of Theorem 1. It is sufficient to check that the Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_d)$ with $d = 1, 2, 7, 11$ satisfy the hypotheses of Theorems 2 (*a*), 2 (*b*), and this is now an easy matter. The only point which requires explanation is that the epimorphisms τ constructed in Lemmas 4·2 and 4·3 have the properties required in hypothesis (iv) of Theorems 2 (*a*), 2 (*b*). Let G_1 be as described in Lemma 4·2 if $d = 1$ and write $G_1 = K_d$ if $d = 2, 7$ or 11. In each case, the image T of the map τ has G_1 as a subgroup of index 2; since G_1 is virtually free, so is T, and hence T is conjugacy separable. Since $\tau|_{G_1} = \mathrm{id}$, the map $\tau|_{G_1}$ is certainly injective. The set $S = \{g \in G \mid gHg^{-1} \leqslant G_1\}$ is clearly a union of cosets $G_1 g$; it contains G_1 and u, and hence

$$\tau(S) \supseteq \tau(G_1) \cup \tau(G_1 u) = G_1 \cup G_1\tau(u).$$

Since $\tau(u) \notin G_1$ and $|T\colon G_1| = 2$ it follows that $\tau(S) = T$. Therefore the hypotheses of Theorems 2 (*a*), 2 (*b*) hold, and the Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_d)$ with $d = 1, 2, 7, 11$ are conjugacy separable.

REFERENCES

[**1**] R. G. Burns. A note on free groups. *Proc. Amer. Math. Soc.* **23** (1969), 14–17.

[**2**] J. L. Dyer. Separating conjugates in free-by-finite groups. *J. London Math. Soc.* (2) **20** (1979), 215–222.

[**3**] J. L. Dyer. Separating conjugates in amalgamated free products and HNN extensions. *J. Austral. Math. Soc. Ser. A* **29** (1980), 35–51.

[**4**] B. Fine. *Algebraic theory of the Bianchi groups* (Marcel Dekker, 1989).

[**5**] E. Formanek. Conjugacy separability in polycyclic groups. *J. Algebra* **42** (1976), 1–10.

[**6**] F. Grunewald and J. Schwermer. Free non-abelian quotients of SL_2 over orders of imaginary quadratic number fields. *J. Algebra* **69** (1981), 162–175.

[**7**] M. Hall. Coset representations in free groups. *Trans. Amer. Math. Soc.* **67** (1949), 421–432.

[**8**] A. G. Howson. On the intersection of finitely generated free groups. *J. London Math. Soc.* **29** (1954), 428–434.

[**9**] N. Jacobson. *Structure of rings* (American Mathematical Society, 1964).

[**10**] A. Karrass, A. Pietrowski and D. Solitar. Finite and infinite cyclic extensions of free groups. *J. Austral. Math. Soc.* **16** (1973), 458–466.

[**11**] J. C. Lennox and J. S. Wilson. On products of subgroups in polycyclic groups. *Arch. Math.* (*Basel*) **33** (1979/80), 305–309.

[**12**] A. Lubotsky. Free quotients and the congruence kernel of SL_2, *J. Algebra* **77** (1982), 411–418.

[**13**] R. C. Lyndon and P. E. Schupp. *Combinatorial group theory* (Springer-Verlag, 1977).

[**14**] A. I. Mal'cev. On homomorphisms onto finite groups. *Uchen. Zap. Ivanovskogo Gos. Ped. Inst.* **18** (1958), 49–60.

[**15**] G. A. Niblo. Separability properties of free groups and surface groups. *J. Pure Appl. Algebra* **78** (1992), 77–84.

[**16**] V. N. Remeslennikov. Groups that are residually finite with respect to conjugacy. *Siberian Math. J.* **12** (1971), 783–792.

[**17**] L. Ribes, D. Segal andP. A. Zalesskii. Conjugacy separability of amalgamated free products with cyclic amalgamation. *J. London Math. Soc.*, to appear.

[**18**] L. Ribes and P. A. Zalesskii. On the profinite topology of a free group. *Bull. London Math. Soc.* **25** (1993), 37–43.

[**19**] L. Ribes and P. A. Zalesskii. Conjugacy separability of amalgamated free products of groups. *J. Algebra* **179**, 751–774 (1996).

[**20**] J.-P. Serre. Le problème des groupes de congruence pour SL_2. *Ann. of Math.* (2) **92** (1970), 489–527.

[**21**] J.-P. Serre. *Trees* (Springer-Verlag, 1980).

[**22**] P. A. Zalesskii. Free constructions of profinite groups and the congruence kernel in case of positive characteristic. *Izv. Ross. Akad. Nauk.* **59** (1995), 59–76.

[**23**] P. A. Zalesskii and O. V. Melnikov. Subgroups of profinite groups acting on trees. *Math. Sb.* **63** (1989), 405–424.

[**24**] P. A. Zalesskii and O. V. Melnikov. Fundamental groups of graphs of profinite groups. *Leningrad Math. J.* **1** (1990), 921–940.