Theorem 11: Profinite HNN-constructions

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Abstract

A more detailed proof of Theorem 11 [3] is given. Also it is emphasized that $\mathrm{HNN}^{abs}(H, \mathcal{A}, \mathcal{B}, T)$ is residually $\mathcal{C}$ when $\mathcal{C}$ is closed under taking products, subgroups and extensions, provided $\mathrm{HNN}(H, \mathcal{A}, \mathcal{B}, T)$ is proper.

1 Introduction

Let us recall the basic situation from [3]. We consider a pro-$\mathcal{C}$ analogue of the concept of an HNN-extension, (cf. [4], p. 180), by generalizing the concept of pro-$\mathcal{C}$ HNN-extension as described in 9.4 of [5]. Following R. Bieri [1] we shall term it a *pro-$\mathcal{C}$* HNN-*group*.

Definition 1 Let H be a pro-$\mathcal{C}$ group and $\partial_0, \partial_1 : (\mathcal{G}, T) \to H$ fiber monomorphisms. A specialization into K consists of a homomorphism $\beta : H \longrightarrow K$ and a continuous map $\beta_1 : T \longrightarrow K$ such that for all $t \in T$ and $g \in \mathcal{G}(t)$ the equality $\beta(\partial_0(h)) = \beta_1(t)^{-1}\beta(\partial_1(h))\beta_1(t)$ is valid. We denote this situation by writing $(\beta, \beta_1) : (H, \mathcal{G}, T) \to K$.

The pro-$\mathcal{C}$ HNN-group is then a pro-$\mathcal{C}$ group G together with a specialization $(\upsilon, \upsilon_1) : (H, \mathcal{G}, T) \longrightarrow G$, with the following universal property: for every pro-$\mathcal{C}$ group K and every specialization $(\beta, \beta_1) : (H, \mathcal{G}, T) \to K$, there exists a unique homomorphism

$$\omega : G \longrightarrow K,$$

such that $\omega\upsilon_1 = \beta_1$ and $\beta = \omega\upsilon$. We shall denote G by $\mathrm{HNN}_{\mathcal{C}}(H, \mathcal{G}, T)$ or simply by $\mathrm{HNN}(H, \mathcal{G}, T)$ when there is no danger of confusion.

Let us compare our definition with [4], p.180 for injective β_1: First, H is the *base group*. Setting $A_t := \partial_0(\mathcal{G}(t))$ and $B_t := \partial_1(\mathcal{G}(t))$, a family $f := \{f_t :\mid t \in T\}$ of isomorphisms is induced setting $f_t(a_t) := \partial_1(g_t)$ for the unique $g_t \in \mathcal{G}(t)$ with $a_t = \partial_0(g_t)$. Thus, the family f satisfies $f_t(a_t) = a_t^t$ for all $a_t \in A_t$ and $t \in T$, and T plays the role of a space of *stable letters*. In fact, below we shall make use of the abstract HNN-group, and denote it by $\mathrm{HNN}^{abs}(H, \mathcal{A}, f, T)$. For T a singleton set, identifying $\mathcal{G}(t)$ with its image under ∂_0 and setting $f := \partial_1$, the definition of a pro-$\mathcal{C}$-HNN extension given in 9.4 in [5] is recovered.

Proposition 2 *Let H be a pro-$\mathcal{C}$ group, $(\mathcal{G}, T)$ a sheaf of pro-$\mathcal{C}$ groups and $\partial_0, \partial_1 : (\mathcal{G}, T) \to H$ fiber monomorphisms. Then there exists a unique pro-$\mathcal{C}$ HNN-group $G = \mathrm{HNN}(H, \mathcal{G}, T)$.*

Proof: This is Proposition 9 [3]. □

A pro-$\mathcal{C}$ HNN-group is a special case of the fundamental pro-$\mathcal{C}$ group $\Pi_1(\mathcal{G}, \Gamma)$ of a profinite graph of pro-$\mathcal{C}$ groups $(\mathcal{G}, \Gamma)$ as introduced in [8]. Namely, a pro-$\mathcal{C}$ HNN-group can be thought as $\Pi_1(\mathcal{G}, \Gamma)$, where Γ is a bouquet (i.e., a connected profinite graph having just one vertex – an isolated point of Γ – that serves as a maximal subtree). Note that *acyclicity* and *simply connectivity* do not coincide in the pro-$\mathcal{C}$ situation, though they do when $\mathcal{C}$ consists of soluble groups only. The pro-$\mathcal{C}$ analogue of a maximal subtree is a maximal $\mathcal{C}$-simply connected subgraph. In general a maximal $\mathcal{C}$-simply connected subgraph in a connected profinite graph might not exist. When it exists the definition of the fundamental pro-$\mathcal{C}$ group $\Pi_1(\mathcal{G}, \Gamma)$ of a graph of pro-$\mathcal{C}$ groups can be given along the lines of the abstract situation as has been done in [7], Section 3, for finite Γ.

Let us recall Lemma 10 [3].

Lemma 3 *Let $G = \mathrm{HNN}(H, \mathcal{G}, T)$ be a pro-$\mathcal{C}$ HNN-group and U an open subgroup of G such that $U \cap \upsilon(\mathcal{G}(t)) = 1$ for all $t \in T$. Then U is a free pro-$\mathcal{C}$ product of conjugates $U \cap \upsilon(H)^g$, for certain $g \in G$ and a free pro-$\mathcal{C}$ group. In particular, if $U \cap \upsilon(H)$ is free, then so is U.*

2 Embedding

We shall need the following criterion for *embedding* a pro-$\mathcal{C}$ group H as a base group into a pro-$\mathcal{C}$ HNN-group $\mathrm{HNN}(H, \mathcal{G}, T)$, whose proof is based on Zoé Chatzidakis' ideas [2]. Let $\partial_0, \partial_1 : (\mathcal{G}, T) \to H$ be fiber monomorphisms, where the restriction of ∂_0 to A_t is the identity. Recall the family f of isomorphisms $f_t : A_t \to B_t$ as described in connection with Definition 1 and let us write φ for υ. If V is an open normal subgroup of H with $f_t(A_t \cap V) = B_t \cap V$ we write $G_V^{abs} := \mathrm{HNN}^{abs}(H/V, \mathcal{A}_V, f_V, T)$ for the abstract HNN-group, where $A_{tV} = A_tV/V$, $B_{tV} = B_tV/V$ are associated subgroups with isomorphisms $f_{tV} : A_tV/V \longrightarrow B_tV/V$ induced by f_t (and we use this notation omitting V if V is trivial). We also shall use the natural injection $\upsilon_{1V}^{abs} : T \to \mathrm{HNN}^{abs}(H/V, \mathcal{A}_\upsilon, T)$ arising from the abstract situation and let $\varphi^{abs} : H \to \mathrm{HNN}^{abs}(H, \mathcal{G}, T)$ denote the canonical embedding.

Notation 4 Given a sheaf morphism $f : (\mathcal{A}, T) \to (\mathcal{B}, T)$ of pro-$\mathcal{C}$ groups all contained in a pro-$\mathcal{C}$ base group H, we let $\mathcal{N}(f)$ denote the filter of all normal subgroups N of G^{abs} with

(i) $G^{abs}/N \in \mathcal{C}$ and

(ii) $({\upsilon_1}^{abs})^{-1}(gN) \cap T$ is clopen in T for all $t \in T$ and all $g \in G^{abs}$.

The universal property of $G^{abs} = \mathrm{HNN}^{abs}(G, \mathcal{G}, T)$ gives rise to a group homomorphism $\lambda : G^{abs} \to G := \mathrm{HNN}(G, \mathcal{G}, T)$ with $\upsilon_1 = \lambda {\upsilon_1}^{abs}$ and $\varphi = \lambda\varphi^{abs}$. We shall identify H and T with their respective images in G^{abs}.

Lemma 5 *For finite H we have $\ker(\lambda) = \bigcap \mathcal{N}(f)$ and $\ker\varphi = \bigcap \mathcal{N}(f) \cap H$.*

Proof:

Claim 1: *Fix $N \in \mathcal{N}(f)$ and let $p^{abs} : G^{abs} \to Q := G^{abs}/N \in \mathcal{C}$ denote the canonical epimorphism. Equip Q with the discrete topology.*
Then there is a continuous epimorphism $p : G \to Q$ with $p\lambda = p^{abs}$.

Property (ii) of N implies that $p^{abs}{\upsilon_1}^{abs} : T \to Q$ is continuous. Let us next show that $p^{abs}\varphi^{abs}\partial_i : \mathcal{G} \to Q$ is continuous. Since ∂_i is continuous, it suffices to ensure that $p^{abs}\varphi^{abs} : H \to Q$ is continuous. However, if (h_ν) is a convergent net in the finite group H, one can pass to a cofinal constant subnet, i.e., we can assume $h_\nu = h$ for some $h \in H$. Therefore $\lim_\nu p^{abs}\varphi^{abs}(h) = p^{abs}\varphi^{abs}(h)$, so that $p^{abs}\varphi^{abs}$ is indeed continous. By the universal property of G^{abs} the relations $\partial_1(a_t)^{{\upsilon_1}^{abs}(t)} = \partial_0(a_t)$ hold for the respective homomorphic images of t and a_t in Q for all $t \in T$ and $a_t \in A_t$. Therefore the universal property of G yields a continuous epimorphism $p : G \to Q$ with $p\lambda = p^{abs}$.

Claim 2: $\ker\lambda \subseteq \bigcap \mathcal{N}(f)$

Suppose $\lambda(g) = 1$ for some $g \in G^{abs}$. Using the notation of claim 1 for arbitrary $N \in \mathcal{N}(f)$ we must have $p\lambda(g) = p^{abs}(g) = 1$. Therefore $g \in N$ holds for all $N \in \mathcal{N}$ and so $g \in \bigcap \mathcal{N}$.

Claim 3: $\ker\lambda \supseteq \bigcap \mathcal{N}$

Pick $g \in \bigcap \mathcal{N}(f)$. Consider an arbitrary continuous epimorphism $p : G \to Q$ for some $Q \in \mathcal{C}$. The universal property of G^{abs} ensures the existence of $p^{abs} : G^{abs} \to Q$ with $p\lambda = p^{abs}$. It is not hard to see that $\ker p^{abs} \in \mathcal{N}(f)$. Then the image of $\lambda(g)$ in Q is trivial. Since $Q \in \mathcal{C}$ is an arbitrary epimorphic image of G^{abs} we can conclude that $\lambda(g) = 1$, as claimed.

Hence we have $\ker\lambda = \bigcap \mathcal{N}(f)$. Finally let us observe that $\ker\varphi = \ker\lambda\varphi^{abs} = (\lambda\varphi^{abs})^{-1}(1) = (\varphi^{abs})^{-1}(\ker\lambda) = \ker\lambda \cap H$. □

We turn to characterizing $\ker\lambda$ and $\ker\varphi$ for an arbitrary pro-$\mathcal{C}$ group H and a sheaf of group $(\mathcal{G}, T)$. Let $\pi_V : H \to H/V$ be the canonical epimorphism. For $V \in \mathcal{V}$ consider the sheaf $(\mathcal{G}_V, T)$ of subgroups of $H_V := H/V$ defined as $\mathcal{G}_V := \mathcal{G}/\partial_0^{-1}(V)$. For V in $\mathcal{V}$ universal properties ensure the existence of canonical group homomorphisms making the following diagram commutative:

$$
\begin{array}{ccccc}
H & \xrightarrow{\varphi^{abs}} & G^{abs} & \xrightarrow{\lambda} & G \\
\downarrow{\scriptstyle \pi_V} & & \downarrow{\scriptstyle {\mu_V}^{abs}} & & \downarrow{\scriptstyle \mu_V} \\
H_V & \xrightarrow{{\varphi_V}^{abs}} & {G_V}^{abs} & \xrightarrow{\lambda_V} & G_V
\end{array}
$$

Recall that the pro-$\mathcal{C}$ HNN-group $G = \mathrm{HNN}(H, \mathcal{G}, T)$ is *proper*, provided that the natural map $\varphi : H \to G$ is a monomorphism.

Theorem 6 *The pro-$\mathcal{C}$ HNN-group $G = \mathrm{HNN}(H, \mathcal{G}, T)$ is proper if and only if H possesses a filter $\mathcal{V}$ of open normal subgroups for which the following conditions hold:*

(i) For all t in T and all V in $\mathcal{V}$ we have $f_t(A_t \cap V) = B_t \cap V$; moreover $\bigcap \mathcal{V} = \{1\}$;

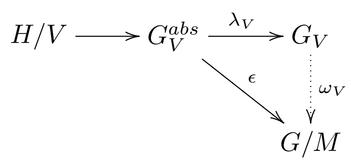
(ii) Having Notation 4 in mind, for every $V \in \mathcal{V}$ the intersection $\bigcap \mathcal{N}(f_V)$ is trivial.

Proof:

When H is finite we may choose $\mathcal{V}$ to contain only the unit group. Then condition (i) is automatically satisfied and the result follows from Lemma 5.

"Properness $\Rightarrow$ (i) & (ii)"

Suppose φ is an embedding. Fix $1 \neq h \in H$. There is $M \triangleleft_o G$ with $G/M \in \mathcal{C}$ and $\varphi(h) \notin M$. Setting $N := \lambda^{-1}(M)$ we find that $V := H \cap N$ will satisfies the first part of property (i). Therefore ${G_V}^{abs}$ is welldefined and universal properties yield a commutative diagram



which implies that $\lambda_V {\varphi_V}^{abs}$ embeds H_V into G_V. Now Lemma 5 shows that $\bigcap \mathcal{N}_V = \{1\}$, and so also (ii) holds.

"(i) & (ii) $\Rightarrow$ properness"

Assume by contradiction that there is $1 \neq h \in H$ and $\lambda(h) = 1$. Property (i) yields $V \in \mathcal{V}$ with $\pi_V(h) \neq 1$. Now property (ii) and the commutativity of the diagram preceding the theorem imply that $\mu_V \lambda(h) = \lambda_V \pi_V(h) \neq 1$, a contradiction. $\square$

Finally let us remark that under the assumptions on $\mathcal{C}$, namely to be closed under products, extensions and subgroups, every abstract free group Φ is residually $\mathcal{C}$. When Φ has rank at least 2 then by Proposition 3.3.15 [5]. Since $\mathcal{C}$ is also subgroup and extension closed also $\mathbf{Z}$ is residually $\mathcal{C}$. Now the argument in the last line of the proof of Theorem 12 on page 806 is valid.

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