PROFINITE TOPOLOGIES IN FREE PRODUCTS OF GROUPS

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1. Introduction

Let H be an abstract group and let $\mathcal{C}$ be a variety of finite groups (i.e., a class of finite groups closed under taking subgroups, quotients and finite direct products); for example the variety of all finite p-groups, for a fixed prime p. Consider the smallest topology on H such that all the homomorphism $H \longrightarrow C$ from H to any group $C \in \mathcal{C}$ (endowed with the discrete topology) is continuous. We refer to this topology as the pro-$\mathcal{C}$ topology of H. This paper is concerned with the following property on H: whenever H_1 and H_2 are finitely generated subgroups of H such that H_1 and H_2 are closed in the pro-$\mathcal{C}$ topology of H, then the subset H_1H_2 of H is closed. If H has this property, we call H "2-product subgroup separable" (relative to the class $\mathcal{C}$; there is an analogous concept of "n-product subgroup separable"). The original motivation for the study of this property goes back to a problem posed by J. Rhodes on the existence of an algorithm to compute the so called kernel of a finite monoid (see [5], [6]). For example, if $\mathcal{C}$ is in addition closed under extensions, then groups that are extensions of free groups by groups in $\mathcal{C}$ are n-product subgroup separable, for any natural number n (see [8], [9]; see also [12] for other examples).

In this paper we show that if the variety $\mathcal{C}$ is closed under extensions, then the property of being 2-product subgroup separable is preserved by taking free products of groups (see Theorem 3.13). This extends in one direction an analogous result of T. Coulbois [1].

The methods used to prove this result are based in the theories of groups acting on trees and of profinite groups acting on profinite trees.

2. Preliminaries

In this paper $\mathcal{C}$ always denotes an extension closed variety of finite groups, i.e., a nonempty collection of finite groups such that

(a) $\mathcal{C}$ is subgroup closed: whenever $G \in \mathcal{C}$ and $H \leq G$, then $H \in \mathcal{C}$;

(b) $\mathcal{C}$ is closed under taking quotients: whenever $G \in \mathcal{C}$ and $K \triangleleft G$, then $G/K \in \mathcal{C}$;

(c) $\mathcal{C}$ is extension closed: whenever $1 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 1$ is an exact sequence of finite groups and $H, K \in \mathcal{C}$, then $G \in \mathcal{C}$.

For example, $\mathcal{C}$ could be the class of all finite groups, or the class of all finite p-groups (for a fixed prime number p), or the class of all finite solvable groups.

All groups considered in this paper are assumed to be residually $\mathcal{C}$ (recall that a group R is residually $\mathcal{C}$ if for any $1 \neq x \in R$, there exists a normal subgroup N of finite index in R such that $x \notin N$ and $G/N \in \mathcal{C}$). It is well-known that an abstract free group is residually $\mathcal{C}$ (see, for example, [7], Proposition 3.3.15), and that a free product of residually $\mathcal{C}$ groups is residually $\mathcal{C}$ (see [4]).

A *pro-$\mathcal{C}$ group* A is an inverse limit

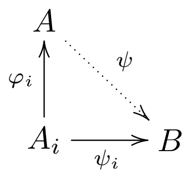
$$A = \varprojlim_{i \in I} A_i$$

of groups in $\mathcal{C}$; we think of G as a topological group with the topology determined by assigning to each finite group A_i the discrete topology. Equivalently, A is a pro-$\mathcal{C}$ group if it is a compact, Hausdorff and totally disconnected topological group such that $A/U \in \mathcal{C}$ for every open normal subgroup U of A. (See [7], Section 9, for general facts about pro-$\mathcal{C}$ groups.)

Let $\{A_1, \ldots, A_n\}$ be a finite collection of pro-$\mathcal{C}$ groups. A free *pro-$\mathcal{C}$ product* of these groups consists of a pro-$\mathcal{C}$ group, denoted $A = \coprod_{i=1}^{n} A_i$, and continuous homomorphisms

$$\varphi_i : A_i \longrightarrow A \quad (i = 1, ..., n),$$

satisfying the following universal property:



for any pro-$\mathcal{C}$ group B and any set of continuous homomorphisms $\psi_i : A_i \longrightarrow B$ $(i = 1, ..., n)$, there exists a unique continuous homomorphism $\psi : A \longrightarrow B$ such that $\psi_i = \psi\varphi_i$, for all $i = 1, ..., n$. Observe that one needs to test the above universal property only for groups $B \in \mathcal{C}$, for then it holds automatically for any pro-$\mathcal{C}$ group B, since such a B is an inverse limit of groups in $\mathcal{C}$. Denote by

$$L = A_1 * \cdots * A_n$$

the free product of $A_1, \ldots, A_n$ as abstract groups. Then $A = \coprod_{i=1}^{n} A_i$ is the completion of L

$$\varprojlim_{N \in \mathcal{N}} L/N,$$

where $\mathcal{N} = \{N \mid N \cap A_i \text{ is open in } A_i \ (i = 1, \ldots, n)\}$. One checks that L is naturally embedded in A (see, for example, [7], Proposition 9.1.8).

Recall that a topological space X is a *profinite space* if it is the inverse limit of finite discrete spaces; in other words, X is profinite if it is compact, Hausdorff and totally disconnected.

A *profinite graph* Γ (oriented) is a profinite space with a distinguish closed subset $V(\Gamma)$ (the vertices of the graph) and a pair of continuous maps $d_0, d_1 : \Gamma \to V(\Gamma)$ (the incidence maps) such that $d_i(v) = v$ for all $v \in V(\Gamma)$ $(i = 0, 1)$. The elements of the subspace $E(\Gamma) = \Gamma - V(\Gamma)$ are the edges of the graph. In this paper, the space $E(\Gamma)$

is assumed to be always closed, and so it is enough to define d_0 and d_1 continuously on $E(\Gamma)$. Let $\mathbf{Z}_{\hat{\mathcal{C}}}$ denote the free pro-$\mathcal{C}$ group of rank 1, and for a profinite space X, let $[\![\mathbf{Z}_{\hat{\mathcal{C}}}X]\!]$ denote the free $\mathbf{Z}_{\hat{\mathcal{C}}}$-module on the basis X (or, equivalently, the free abelian pro-$\mathcal{C}$ group on the basis X). Such a profinite graph is called a *pro-$\mathcal{C}$ tree* if the following sequence

$$0 \longrightarrow [\![\mathbf{Z}_{\hat{\mathcal{C}}}E(\Gamma)]\!] \stackrel{\delta}{\longrightarrow} [\![\mathbf{Z}_{\hat{\mathcal{C}}}V(\Gamma)]\!] \stackrel{\varepsilon}{\longrightarrow} \mathbf{Z}_{\hat{\mathcal{C}}} \longrightarrow 0$$

of free pro-$\mathcal{C}$ abelian groups is exact, where $\varepsilon(v) = 1$ for every $v \in V(\Gamma)$, $\delta(e) = d_1(e) - d_0(e)$ for every $e \in E(\Gamma)$. (See [3] for a general definition of pro-$\mathcal{C}$ tree and its properties.) Finite abstract graphs are profinite graphs; and finite abstract trees are pro-$\mathcal{C}$ trees for any $\mathcal{C}$.

Let Γ be a profinite pro-$\mathcal{C}$ tree and let $x, y \in \Gamma$. The *geodesic* $[x, y]$ determined by x and y is the smallest profinite subtree of Γ containing x and y, or equivalently, the intersection of all profinite subtrees of Γ containing x and y.

If Γ and Γ' are profinite graphs, a *morphism* $\alpha : \Gamma \longrightarrow \Gamma'$ is simply a continuous map such that $\alpha(d_i(x)) = d_i(\alpha(x))$, for all $x \in \Gamma$ $(i = 0, 1)$. A morphism is an *embedding* if it is an injection.

Let A be a profinite group. We say that A acts on a profinite graph Γ from the left if there exists a continuous function $A \times \Gamma \longrightarrow \Gamma$, denoted $(a, x) \mapsto ax$ $(a \in A, x \in \Gamma)$, such that $(aa')x = a(a'x)$, $1x = x$ and $d_i(ax) = ad_i(x)$, for all $a, a' \in G, x \in \Gamma$ $(i = 0, 1)$. There is a similar concept of right action of A on Γ. If a profinite group A acts from the left on a profinite graph Γ, we denote the corresponding quotient graph of orbits by $A\backslash\Gamma$. If A acts on Γ from the right, we denote the quotient graph by Γ/A. Let A act on Γ from the left and let

$$\varphi : \Gamma \longrightarrow A\backslash\Gamma$$

be the corresponding quotient map. If $A\backslash\Gamma$ is finite, there is a maximal subtree T' of $A\backslash\Gamma$; hence there exists a *connected φ-transversal* (also called simply a transversal) J containing a lifting T of T', i.e., T is a subtree of Γ that is mapped isomorphically to T' by φ, J is a subset (not necessarily a subgraph) of Γ containing T such that φ induces a bijection from J to $A\backslash\Gamma$, $d_0(J) \subseteq J$ and $V(\Gamma) \cap J = V(T)$.

Let $A = \coprod_{i=1}^n A_i$ be a free pro-$\mathcal{C}$ product. Then the *standard pro-$\mathcal{C}$ tree* $S(A)$ associated with this free product is defined as follows (cf. [3]): its space of edges is the disjoint union

$$E(S(A)) = \dot{\bigcup}_{i=1}^{n} A$$

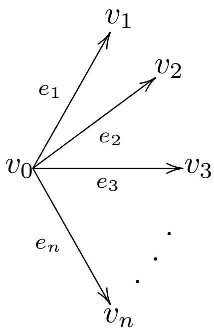
of n copies of A; its space of vertices is the disjoint union

$$V(S(A)) = \dot{\bigcup}_{i=0}^{n} A/A_i,$$

of the quotient spaces A/A_i, where $A_0 = 1$; and its incidence maps d_0 and d_1 are given by

$$d_0(a) = aA_0 = a, \; d_1(a) = aA_i, \text{ when } a \text{ is in the } i\text{th copy of } A \text{ in } E(S(A)) = \dot{\bigcup}_{i=1}^{n} A.$$

Note that A acts naturally on $S(A)$ by left multiplication; the stabilizers of vertices are conjugates of the groups A_i $(i = 1, \ldots, n)$, and all edge stabilizers are trivial. The quotient graph $A\backslash S(A)$ is a finite tree T_n with n edges and $n + 1$ vertices:



Using this finite graph T_n, one can give an alternative description of $S(A)$: we identify T_n with a canonical transversal of it in $S(A)$ whose vertices are $v_i = 1A_i$ $(i = 0, 1, \ldots, n)$; then $S(A)$ is the unique profinite graph obtained as the union of all translations of T_n by the elements of the group A, with the proviso that the A-stabilizer of the vertex av_i is aA_ia^{-1} $(a \in A, i = 0, 1, \ldots, n)$ and the A-stabilizer of the edge ae_i is trivial $(i = 1, \ldots, n)$; furthermore, the topology of $S(A)$ is induced by the product topologies of $A \times V(T_n)$ and $A \times E(T_n)$.

Remark that if B_i is a closed subgroup of A_i $(i = 1, \ldots, n)$ and if $B = \coprod_{i=1}^n B_i$ is the free pro-$\mathcal{C}$ product of the pro-$\mathcal{C}$ groups $B_1, \ldots, B_n$, then B is the closed subgroup of A generated by $B_1, \ldots, B_n$ (cf. [3], Corollary 9.1.7); hence there is a natural embedding of the corresponding pro-$\mathcal{C}$ graphs $S(B) \hookrightarrow S(A)$.

There is a similar construction of an abstract tree $S(G)$ associated with a free product $G = G_1 * \cdots * G_n$ of abstract groups $G_1, \ldots, G_n$ (cf. [10], Section 4.5). Its space of edges is the disjoint union $E(S(G)) = \dot\bigcup_{i=1}^{n} G$ of n copies of G; its space of vertices is the disjoit union $V(S(G)) = \dot\bigcup_{i=0}^{n} G/G_i$, of the quotient spaces G/G_i, where $G_0 = 1$; and its incidence maps d_0 and d_1 are given by $d_0(g) = gG_0 = g$ and $d_1(g) = aG_i$, when $g \in E(S(G))$ is in the ith copy of G in $\mathrm{E(S(G))} = \dot\bigcup_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{G}$.

The group G acts naturally on $S(G)$ by left multiplication, and the corresponding quotient graph is the above finite tree T_n with n edges and $n+1$ vertices.

Let R be an abstract group. Denote by $\mathcal{N}_\mathcal{C}$ the collection of all normal subgroups N of R such that $R/N \in \mathcal{C}$. Then there is a unique topology on R making it into a topological group such that $\mathcal{N}_\mathcal{C}$ is a fundamental system of neighborhoods of the identity element 1 of R. This is the *(full) pro-$\mathcal{C}$ topology of R* . We say that R is *n-product subgroup separable* (with respect to its pro-$\mathcal{C}$ topology) if whenever $H_1, \ldots, H_n$ are closed subgroups of R (in the pro-$\mathcal{C}$ topology of R) which are finitely generated as abstract groups, then the product subset $H_1 \cdots H_n$ is closed in the pro-$\mathcal{C}$ topology of R.

If R is an abstract group, we denote by $R_{\hat{\mathcal{C}}}$ its completion with respect to its pro-$\mathcal{C}$ topology, i.e.,

$$R_{\hat{\mathcal{C}}} = \varprojlim_{N \in \mathcal{N}_\mathcal{C}} R/N.$$

Then, there exists a natural embedding

$$\iota : R \longrightarrow R_{\hat{\mathcal{C}}},$$

since R is assumed to be residually $\mathcal{C}$.

The following fact is not hard to prove: 'pro-$\mathcal{C}$ completion commutes with free products', in other words, if

$$G = G_1 * \cdots * G_n$$

is a free product of abstract groups $G_1, \ldots, G_n$, then

$$G_{\hat{\mathcal{C}}} = (G_1)_{\hat{\mathcal{C}}} \amalg \cdots \amalg (G_n)_{\hat{\mathcal{C}}} \quad \text{(free pro}-\mathcal{C}\text{ product).}$$

Furthermore, since the groups G_i $(i = 1, \ldots, n)$ are assumed to be residually $\mathcal{C}$, so is their free product G, and we have canonical embeddings

$$\begin{array}{ccc} G_i & \hookrightarrow & (G_i)_{\hat{\mathcal{C}}} \\ \downarrow & & \downarrow \\ G & \hookrightarrow & G_{\hat{\mathcal{C}}} \end{array}$$

Moreover, each G_i is closed in the pro-$\mathcal{C}$ topology of G (cf. [7], Corollary 3.1.6).

These facts allow us to think of the tree $S(G)$, associated with the abstract free product $G = G_1 * \cdots * G_n$, as a subgraph of the pro-$\mathcal{C}$ tree $S(G_{\hat{\mathcal{C}}})$, associated with the free pro-$\mathcal{C}$ product $G_{\hat{\mathcal{C}}} = (G_1)_{\hat{\mathcal{C}}} \amalg \cdots \amalg (G_n)_{\hat{\mathcal{C}}}$. More precisely, there is a natural embedding of graphs

$$S(G) \longrightarrow S(G_{\hat{\mathcal{C}}})$$

defined as follows.

For vertices: $gG_i \mapsto g(G_i)_{\hat{\mathcal{C}}} \quad (g \in G, i = 1, \ldots, n)$
For edges: g in the i-th copy of G in $E(S(G))$ is sent to
g in the i-th copy of $G_{\hat{\mathcal{C}}}$ in $E(S(G_{\hat{\mathcal{C}}}))$.

Notation: if H is a subgroup of a group G and $x, y \in G$, then as usual, $y^x = x^{-1}yx$ and $H^x = x^{-1}Hx$. If X is a subset of a group G, then $\bar{X}$ denotes the closure of X in the pro-$\mathcal{C}$ completion $G_{\hat{\mathcal{C}}}$ of G; observe that the closure $Cl(X)$ of X in the pro-$\mathcal{C}$ topology of G coincides with $G \cap \bar{X}$.

3. The Main Theorem

We begin with a reduction result.

Lemma 3.1 Let R be an abstract group, endowed with its pro-$\mathcal{C}$ topology, and let U be an open subgroup of R. Then R is n-product subgroup separable if and only if U is n-product subgroup separable.

Proof: First observe that since $\mathcal{C}$ is extension closed, the pro-$\mathcal{C}$ topology of U is precisely the topology induced by the pro-$\mathcal{C}$ topology of R (see Lemma 3.1.4(a) in [7]). Assume R is n-product subgroup separable. Then plainly U is n-product subgroup separable.

Conversely assume that U is n-product subgroup separable. By the above, the core U_R of U in R is n-product subgroup separable as well. Hence, replacing U by U_R, if necessary, we may assume that U is open and normal in R. Let $H_1, \ldots, H_n$ be finitely generated closed subgroups of R. We shall prove by induction on the number of H_i which are not contained in U that $H_1 \cdots H_n$ is closed in the pro-$\mathcal{C}$ topology of R. If $H_i \leq U$ for all $i = 1, \ldots, n$, the result is clear. Since each H_i is finitely generated and $U \cap H_i$ has finite index in H_i, we have that $U \cap H_i$ is also finitely generated. Pick $H_t \not\leq U$. Write $H_t = \dot{\bigcup}_j h_j(U \cap H_t)$, $(h_j \in H_t)$. Therefore we get a finite union

$$H_1 \cdots H_n = \dot{\bigcup}_j h_j H_1^{h_j} \cdots H_{t-1}^{h_j}(U \cap H_t)H_{t+1} \cdots H_n.$$

By the induction hypothesis, $H_1^{h_j} \cdots H_{i-1}^{h_j}(U \cap H_i)H_{i+1} \cdots H_n$ is closed in R. Thus $H_1 \cdots H_n$ is closed in R. □

Lemma 3.2 Let G be an abstract group that acts freely on a tree T. Endow G with its pro-$\mathcal{C}$ topology. Let K be a closed subgroup of G and let Δ be a finite subgraph of $K\backslash T$. Then there exists an open subgroup V of G containing K such that the natural map of graphs

$$\tau_V : K\backslash T \longrightarrow V\backslash T$$

is injective on Δ.

Proof: Since K is closed, $K = \bigcap_{i \in I} U_i$, where $\{U_i \mid i \in I\}$ is the collection of all open subgroups of G containing K. Consider the map of graphs $\tau_G : K\backslash T \longrightarrow G\backslash T$. Since Δ is finite, it is a finite union of intersections as follows:

$$\Delta = \bigcup_{t=1}^{m} (\Delta \cap \tau_G^{-1}(x_t)),$$

for some $x_t \in G\backslash T$ and $m \in \mathbf{N}$. We claim that for each $t = 1, \ldots, m$, there exists some $i_t \in I$ such that $\tau_{U_{i_t}}$ is injective on $\Delta \cap \tau_G^{-1}(x_t)$. Since G acts freely on T, the set $\tau_G^{-1}(x_t)$ may be identified with $K\backslash G$; moreover, if $K \leq U \leq G$, the restriction of $\tau_U : K\backslash T \longrightarrow U\backslash T$ to $\tau_G^{-1}(x_t)$ may be identified with the canonical surjection $\tau_U : K\backslash G \longrightarrow U\backslash G$. Since $\Delta \cap \tau_G^{-1}(x_t)$ can be thought of as a finite subset of $K\backslash G$, the existence of the required i_t follows from $K = \bigcap_{i \in I} U_i$. Define V to be $V = \bigcap_{t=1}^{m} U_{i_t}$. Then clearly τ_V is injective on Δ. □

Let v be a vertex of an abstract graph Γ. Then $Star_\Gamma(v)$ is the set of edges e of Γ such that $v = d_0(e)$ or $v = d_1(e)$. A morphism of graphs $\varphi : \Gamma \longrightarrow \Gamma'$ is called an *immersion* if, for each vertex $v \in \Gamma$, the map $Star_\Gamma(v) \longrightarrow Star_{\Gamma'}(\varphi(v))$, induced by φ, is an injection.

Lemma 3.3 Let $\varphi : \Gamma \longrightarrow \Delta$ be an immersion of finite connected graphs. If φ induces an epimorphism $\pi_1(\Gamma) \longrightarrow \pi_1(\varphi(\Gamma))$, then φ is an injection.

Proof: An immersion of graphs induces a monomorphism of fundamental groups (cf. Proposition 5.3 in [11]); hence, $\pi_1(\Gamma) \longrightarrow \pi_1(\varphi(\Gamma))$ is an isomorphism. Define m to be the rank of the free group $\pi_1(\Gamma)$. Then we also have rank$(\pi_1(\varphi(\Gamma)) = m$. By Corollary 1.8 in [2] we have that

$$\sum_{v\in V(\Gamma)} (|Star_\Gamma(v)| - 2) = 2m - 2 = \sum_{w\in V(\varphi(\Gamma))} (|Star_{\varphi(\Gamma)}(w)| - 2).$$

Since $|Star_\Gamma(v)| \geq |Star_{\varphi(\Gamma)}(\varphi(v))|$ for all $v \in V(\Gamma)$, we deduce that

$$|Star_\Gamma(v)| = |Star_{\varphi(\Gamma)}(\varphi(v))|, \quad \forall v \in V(\Gamma), \quad \text{and} \quad |V(\Gamma)| = |V(\varphi(\Gamma))|.$$

I.e, φ is an injection. □

Lemma 3.4 Let an abstract group G act on an abstract tree S. Let K be a closed subgroup (in the pro-$\mathcal{C}$ topology) of G. Let D be a K-invariant subtree of S such that $K\backslash D$ is finite. Endow G with its pro-$\mathcal{C}$ topology; then for every open subgroup U of G containing K, there exists an open subgroup V of G with $K \leq V \leq U \leq G$ such that the morphism

$$\tau_V : K\backslash D \longrightarrow V\backslash S$$

induces an epimorphism of fundamental groups

$$f_V : \pi_1(K\backslash D) \longrightarrow \pi_1(\tau_V(K\backslash D)).$$

Proof: Let $\tilde{K}$ denote the subgroup of K generated by all the stabilizers of the vertices of D (under the action of K) and let $\tilde{U}$ denote the subgroup of U generated by all stabilizers of the vertices of S (under the action of U); observe that $\tilde{K} \triangleleft K$ and $\tilde{U} \triangleleft U$. Then $K/\tilde{K} = \pi_1(K\backslash D)$ and $U/\tilde{U} = \pi_1(U\backslash S)$ (cf. [10], page 55, Corollary 1). Moreover $\tilde{U}\backslash S$ is a tree (see [10], page 55, Exercise 2); since $U/\tilde{U}$ acts freely on this tree, it follows that $U/\tilde{U}$ is a free group. Consider the image D_U of D in $\tilde{U}\backslash S$. Clearly $K\tilde{U}/\tilde{U}$ acts freely on D_U and hence, $\pi_1((K\tilde{U}/\tilde{U})\backslash D_U) = K\tilde{U}/\tilde{U}$ (use again [10], page 55, Corollary 1). Therefore, since $\tilde{K} \leq \tilde{U}$, the homomorphism

$$f_{K\tilde{U}} : \pi_1(K\backslash D) \longrightarrow \pi_1(\tau_{\tilde{U}K}(K\backslash D)) = \pi_1((K\tilde{U}/\tilde{U})\backslash D_U)$$

coincides with the natural epimorphism $K/\tilde{K} \longrightarrow K\tilde{U}/\tilde{U}$.

Since $U/\tilde{U}$ is free and acts freely on $\tilde{U}\backslash S$, by Lemma 3.2 there exists an open subgroup V of U containing $K\tilde{U}$ such that the restriction

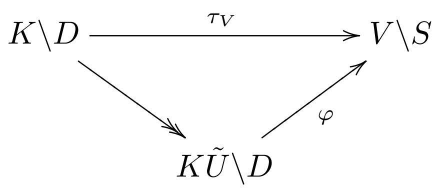
$$\varphi : (K\tilde{U}/\tilde{U})\backslash D_U \longrightarrow (V/\tilde{U})\backslash(\tilde{U}\backslash S(G))$$

of the natural morphism

$$(K\tilde{U}/\tilde{U})\backslash(\tilde{U}\backslash S) \longrightarrow (V/\tilde{U})\backslash(\tilde{U}\backslash S)$$

to $(K\tilde{U}/\tilde{U})\backslash D_U$ is an injection.

Clearly $(V/\tilde{U})\backslash(\tilde{U}\backslash S(G)) = V\backslash S(G)$ and $(K\tilde{U}/\tilde{U})\backslash D_U = K\tilde{U}\backslash D$. Hence from the commutativity of the diagram



one deduces that $\varphi((K\tilde{U}/\tilde{U})\backslash D_U) = \tau_V(K\backslash D)$. In other words

$$(K\tilde{U}/\tilde{U})\backslash D_U \longrightarrow \tau_V(K\backslash D)$$

is an isomorphism. So it induces an isomorphism of fundamental groups

$$\eta : \pi_1(K\tilde{U}/\tilde{U})\backslash D_U \longrightarrow \pi_1(\tau_V(K\backslash D)).$$

Thus $f_V = \eta f_{K\tilde{U}}$ is an epimorfism as asserted. $\square$

Lemma 3.5 Let $G_1, \dots, G_m$ be groups and let H be a finitely generated closed subgroup of the free product $G = G_1 * \cdots * G_m$ (endowed with its pro-$\mathcal{C}$ topology). Let $S(G)$ be the standard tree of the free product $G = G_1 * \cdots * G_m$ and let D be an H-invariant subtree of $S(G)$ such that $H\backslash D$ is finite. Then for any connected transversal Σ_H of $H\backslash D$ in D, there exists an open subgroup U of G and a connected transversal Σ_U of $U\backslash S(G)$ in $S(G)$ such that

(a) $H\backslash D$ is canonically embedded in $U\backslash S(G)$ and $\Sigma_H \subseteq \Sigma_U$;

(b)
$$U = \left[\mathop{\ast}_{w\in V(\Sigma_U)} U_w\right] * F_U,$$
where F_U is the free group $\pi_1(U\backslash S(G))$ and where U_w denotes the stabilizer in U of the vertex w;

(c)
$$H = \left[\mathop{\ast}_{w\in V(\Sigma_H)} H_w\right] * F_H,$$
where F_H is the free group $\pi_1(H\backslash D)$ and where H_w is the stabilizer in H of the vertex w;

(d) F_H is a free factor of F_U.

Proof: Consider the canonical morphism of graphs

$$\tau_U : H\backslash D \longrightarrow H\backslash S(G) \longrightarrow U\backslash S(G).$$

Observe first that, by Lemma 3.4, for every open subgroup U containing H there exists an open subgroup $V \le U$ containing H such that the morphism τ_V induces an epimorphism of fundamental groups $f_V : \pi_1(H\backslash D) \longrightarrow \pi_1(\tau_V(H\backslash D)$.

Hence, by Lemma 3.3, to show that the morphism above is injective it suffices to show the existence of an open subgroup U containing H such that τ_U is an immersion. Choose $\bar{w} \in V(H\backslash D)$, where $\bar{w} = Hw$ and $w \in D \subset S(G)$. Since $H\backslash D$ is finite, it suffices to prove the existence of U such that $(\tau_U)_{|Star_{H\backslash D}(\bar{w})}$ is injective. Now we use the structure of $G\backslash S(G)$. If $w = v$, then the result follows from the fact that $|Star_{H\backslash D}(\bar{v})| = |Star_D(v)|$ and $|Star_{U\backslash S(G)}(\bar{v})| = |Star_{S(G)}(v)|$, for every open subgroup U.

Assume next that $\bar{w} \neq vH$. Then $Star_{H\backslash D}(\bar{w})$ is a finite subset of $Star_{H\backslash S(G)}(\bar{w}) = (H \cap G_w)\backslash G_w$. Since H is closed, we have that $H = \bigcap V$, where V ranges over all the open subgroups of G containing H. Hence $H \cap G_w = \bigcap (V \cap G_w)$; so there exists an open subgroup U of G containing H such that all elements of $Star_{H\backslash D(w)}$ are distinct modulo $U \cap G_w$ as needed. This proves that τ_U is injective.

Thus we may regard $H\backslash D$ as a subgraph of $U\backslash S(G)$. Choose a maximal subtree T_H of $H\backslash D$ and extend it to a maximal subtree T_U of $U\backslash S(G)$. Let

$$j : H\backslash D \longrightarrow \Sigma_H$$

be a bijection onto a connected transversal Σ_H of $H\backslash D$ in D containing a lifting of T_H. Extend j to a bijection, which we denote also by j,

$$j : U\backslash S(G) \longrightarrow \Sigma_U$$

onto a connected transversal Σ_U of $U\backslash S(G)$ in $S(G)$ containing a lifting of T_U.

For every edge $e \in U\backslash S(G) - T_U$, choose an element $g_e \in G$ such that

$$g_e j(d_1(e)) = d_1(j(e)).$$

Then (see, for example, [10], Section I.5.5, Theorem 14)

$$U = \mathop{\ast}_{w \in V(\Sigma_U)} U_w * F_U,$$

where F_U is a free group on the set $B_U = \{g_e \mid e \in U\backslash S(G) - T_U\}$ and where U_w is the stabilizer in U of the vertex w.

We also have that

$$H = [\mathop{\ast}_{w \in V(\Sigma_H)} H_w] * F_H,$$

where F_H is a free group on the set $B_H = \{g_e \mid e \in H\backslash D - T_H\}$ and where H_w is the stabilizer in H of the vertex w.

Part (d) is clear since $H\backslash D$ is a subgraph of $U\backslash S(G)$. □

Corollary 3.6 Let $G_1, \ldots, G_m$ be groups and let H be a finitely generated closed subgroup of the free product $G = G_1 * \cdots * G_m$. Then there exists an open subgroup U of G and (Kurosh-type) decompositions

$$U = U_1 * \cdots * U_t \quad \text{and} \quad H = H_1 * \cdots * H_t$$

such that

(a) $H_i \leq_c U_i$ $(i = 1, \ldots, t)$;
(b) For each $i = 1, \ldots, t-1$, U_i is an open subgroup of a conjugate of some G_j $(j = 1, \ldots, m)$, i.e., $U_i = U \cap \tau G_j \tau^{-1}$ for some $\tau \in G$;
(c) U_t is a free group of finite rank and H_t is a free factor of U_t.

Moreover, the decomposition for U (respectively for H) can be chosen to contain as factors all the intersections $U \cap G_i$ (respectively, $H \cap G_i$) $(1 \leq i \leq m)$.

Proof: Consider the standard tree $S(G)$ of the free product $G = G_1 * \cdots * G_m$. Define a subtree of $S(G)$

$$D = \big(\bigcup_{j \in J} H[v_0, r_j v]\big) \cup \big(\bigcup_{i=1}^{m} H[v_0, v_i]\big),$$

where $\{r_j \mid j \in J\}$ is a finite set of generators for H. Choose a connected transversal Σ_H of $H \backslash D$ in D containing all the v_i. Now apply the preceding lemma and observe that if $v \in \Sigma$ and $v = \tau v_j$, then $U_v = U \cap \tau G_j \tau^{-1}$ and $H_v = H \cap \tau G_j \tau^{-1}$; in particular, $U_{v_i} = U \cap G_i$ and $H_{v_i} = H \cap G_i$. □

Corollary 3.7 Let $G_1, \ldots, G_m$ be groups and let $G = G_1 * \cdots * G_m$ be their free product. Assume that H is a finitely generated and closed subgroup of G (in its pro-$\mathcal{C}$ topology). Then there exists a Kurosh decomposition

$$H = \big[\ast_{i=1}^{n} \big[\ast_{\tau \in H \backslash G / G_i} H \cap \tau G_i \tau^{-1}\big]\big] * F$$

of H, where F is a free group, such that

$$\bar{H} = \big[\coprod_{i=1}^{n} \big[\coprod_{\tau \in H \backslash G / G_i} \overline{H \cap \tau G_i \tau^{-1}}\big]\big] \amalg \bar{F},$$

where if X is a subset of H, then $\bar{X}$ denotes the topological closure of X in $G_{\hat{\mathcal{C}}}$. Moreover, $\bar{F} = F_{\hat{\mathcal{C}}}$ is a free pro-$\mathcal{C}$ group.

Proof: Choose U open in G and Kurosh decompositions

$$U = U_1 * \cdots * U_t \quad \text{and} \quad H = H_1 * \cdots * H_t$$

satisfying the conditions of Corollary 3.6. Using the fact that U is open in G and the form of the decomposition, one can show that

$$\bar{U} = \bar{U}_1 \amalg \cdots \amalg \bar{U}_t$$

where $\bar{U} = U_{\hat{\mathcal{C}}}$ and $\bar{U}_t = (U_t)_{\hat{\mathcal{C}}}$ is a free pro - $\mathcal{C}$ group (cf. [7], Corollary 9.1.7 and Theorem 9.1.9). Next observe that $\bar{H}$ coincides with the closed subgroup of $\bar{U}$ generated by the groups $\bar{H}_i$ $(i = 1, \ldots, t)$. Note that the latter group is $\bar{H}_1 \amalg \cdots \amalg \bar{H}_t$ (cf. [7], Corollary

9.1.7). Finally, since H_t is a free factor of U_t, we have that the topology on H_t induced from the pro-$\mathcal{C}$ topology of U_t coincides with the full pro-$\mathcal{C}$ topology of H_t (cf. [7], Corollary 3.1.6); therefore $\bar{F} = F_{\hat{\mathcal{C}}}$. $\square$

Lemma 3.8 Let $G_1, \ldots, G_m$ be groups and let H be a closed subgroup of the free product $G = G_1 * \cdots * G_m$ (endowed with its pro-$\mathcal{C}$ topology). Let $S(G)$ be the standard tree of the free product $G = G_1 * \cdots * G_m$ and let D be an H-invariant subtree of $S(G)$ such that $H \backslash D$ is finite. Then

$$H \backslash D = \bar{H} \backslash \bar{D},$$

where $\bar{H}$ denotes the closure of H in $G_{\hat{\mathcal{C}}}$, and $\bar{D}$ is the closure of D in $S(G_{\hat{\mathcal{C}}})$.

Proof: Consider the natural continuous map

$$D \hookrightarrow \bar{D} \twoheadrightarrow \bar{H} \backslash \bar{D}$$

Since its image is dense and $H \backslash D$ is finite, it induces an onto map

$$H \backslash D \longrightarrow \bar{H} \backslash \bar{D}.$$

Now, by Lemma 3.2, there exists an open subgroup U of G containing H such that

$$\tau : H \backslash D \longrightarrow H \backslash S(G) \longrightarrow U \backslash S(G)$$

is injective. Since U is open, one clearly has $U \backslash S(G) = \bar{U} \backslash S(G_{\hat{\mathcal{C}}})$ (in this case the space edges of these quotient graphs is the set of right cosets $U \backslash G = \bar{U} \backslash \bar{G}$, and the set of vertices is the set of open double cosets $U \backslash G / G_i = \bar{U} \backslash G_{\hat{\mathcal{C}}} / (G_i)_{\hat{\mathcal{C}}}$). From the commutativity of the diagram

$$\begin{array}{ccccccc} K \backslash D & \hookrightarrow & H \backslash S(G) & \longrightarrow & U \backslash S(G) & \xrightarrow{=} & \bar{U} \backslash S(G_{\hat{\mathcal{C}}}) \\ & \searrow & & & & \nearrow & \\ & & \bar{H} \backslash \bar{D} & & & & \end{array}$$

one deduces that $H \backslash D \longrightarrow \bar{H} \backslash \bar{D}$ is injective. $\square$

Lemma 3.9 Let $A = B \amalg C$ be the free pro-$\mathcal{C}$ product of pro-$\mathcal{C}$ groups B and C. Assume that $B_1 \leq_c B$, $C_1 \leq_c C$ and $A = B \amalg C = \overline{\langle B_1, C_1 \rangle}$. Then $B = B_1$ and $C = C_1$.

Proof: Let $\varphi : A \longrightarrow B$ be the epimorphism induced by the identity homomorphism $B \longrightarrow B$ and the homomorphism that sends C to 1. Since A is generated by B_1 and C_1 and since $\varphi(C_1) = 1$, it follows that $B = \varphi(B_1) = B_1$. Similarly $C = C_1$. $\square$

Lemma 3.10 Let $G_1, \ldots, G_m$ be residually $\mathcal{C}$ groups and let

$$G = G_1 * \cdots * G_m.$$

Endow G with the pro-$\mathcal{C}$ topology. Let H be a subgroup of G which is either open or finitely generated and closed. Let

$$L = (G_1)_{\hat{\mathcal{C}}} * \cdots * (G_m)_{\hat{\mathcal{C}}}$$

be the abstract free product of the pro-$\mathcal{C}$ completions of the groups G_i. Denote by D the minimal H-invariant subtree of $S(G)$ containing v_0 if H is finitely generated, and let $D = S(G)$ if H open. Let Σ_H be a connected transversal of $H\backslash D$ in D. Then

$$H = \big[\ast_{v\in V(\Sigma_H)} H_v\big] * \pi_1(H\backslash D) = \big[\ast_{i=1}^{m}\big[\ast_{\tau\in H\backslash G/G_i} H\cap \tau G_i\tau^{-1}\big]\big] * F$$

and

$$\bar{H}\cap L = \big[\ast_{v\in V(\Sigma_H)} \overline{(H_v)}\big] * F = \big[\ast_{i=1}^{m}\big[\ast_{\tau\in H\backslash G/G_i} \overline{H\cap \tau G_i\tau^{-1}}\big]\big] * F,$$

where $F = \pi_1(H\backslash D)$.

Furthermore, for $\tau \in H\backslash G/G_i$ as above, $(i = 1,\dots,m)$,

$$\bar{H}\cap \tau\bar{G}_i\tau^{-1} = \overline{H\cap \tau G_i\tau^{-1}}.$$

Proof: Note that $G_{\hat{\mathcal{C}}} = (G_1)_{\hat{\mathcal{C}}} \amalg \cdots \amalg (G_m)_{\hat{\mathcal{C}}}$. By Lemma 3.8 $H\backslash D = \bar{H}\backslash\bar{D}$. Let D' be the intersection of $\bar{D}$ with the abstract connected component of $S(G_{\hat{\mathcal{C}}})$ containing $S(G)$ (this connected component coincides with $S(L)$). Then

$$(\bar{H}\cap L)\backslash D' = H\backslash D = \bar{H}\backslash\bar{D};$$

indeed, the natural map of graphs

$$D' \longrightarrow \bar{H}\backslash\bar{D}$$

is clearly an epimorphism, and if $d_1 = hd_2$ for some $h\in\bar{H}$, $d_1, d_2 \in D'$, then $h \in \bar{H}\cap L$ (just notice that d_1 and d_2 are either both in L or both of the form gv_i, where $g\in L$, $i = 1,\dots,n$), i.e.,

$$(\bar{H}\cap L)\backslash D' \longrightarrow \bar{H}\backslash\bar{D}$$

is bijective.

Let Σ_H be a connected transversal of $H\backslash D$ in $S(G)$. Put $F = \pi_1(H\backslash D) = \pi_1((\bar{H}\cap L)\backslash D')$. Then (cf. [10], page 43, Example 1)

$$H = \big[\ast_{v\in \Sigma_H} H_v\big] * \pi_1(H\backslash D) = \big[\ast_{i=1}^{m}\big[\ast_{\tau\in H\backslash G/G_i} H\cap \tau G_i\tau^{-1}\big]\big] * F$$

and

$$\begin{aligned}\bar{H}\cap L &= \big[\ast_{v\in \Sigma_H} (\bar{H}\cap L)_v\big] * \pi_1((\bar{H}\cap L)\backslash D')\\ &= \big[\ast_{i=1}^{m}\big[\ast_{\tau\in H\backslash G/G_i} \bar{H}\cap \tau\bar{G}_i\tau^{-1}\big]\big] * F. \qquad (2)\end{aligned}$$

It remains to prove that for $\tau\in H\backslash G/G_i$ as above and $i = 1,\dots,m$,

$$\bar{H}\cap \tau\bar{G}_i\tau^{-1} = \overline{H\cap \tau G_i\tau^{-1}}.$$

Suppose first that H is open. Then $\bar{H} = H_{\hat{\mathcal{C}}}$ and so,

$$\bar{H} = [\coprod_{i=1}^{n} \coprod_{\tau \in H\backslash G/G_i} (H \cap \tau G_i \tau^{-1})_{\hat{\mathcal{C}}}] \amalg F_{\hat{\mathcal{C}}} = [\coprod_{i=1}^{n} \coprod_{\tau \in H\backslash G/G_i} \overline{H \cap \tau G_i \tau^{-1}}] \amalg \bar{F}$$

(see Exercise 9.1.1(a) and Corollary 3.1.6 in [7]). Note that since H is open $H\backslash G/G_i = \bar{H}\backslash G_{\hat{\mathcal{C}}}/G_i$; it follows that (see Theorem 9.1.9 in [7] and its proof together with the equation (2) above)

$$\bar{H} = \coprod_{i=1}^{n} \coprod_{\tau \in H\backslash G/G_i} (\bar{H} \cap \tau \bar{G}_i \tau^{-1}) \amalg F_{\hat{\mathcal{C}}}.$$

Then, comparing these two decompositions of $\bar{H}$ and using Lemma 3.9 we get that $\bar{H} \cap \tau \bar{G}_i \tau^{-1} = \overline{H \cap \tau G_i \tau^{-1}}$.

Suppose now that H is closed and finitely generated. Then $H \cap \tau G_i \tau^{-1}$ is also closed.

Let $\mathcal{V}$ be the set of all open subgroups of G containing H. Then $H = \bigcap_{V \in \mathcal{V}} V$ because H is closed. Hence

$$H \cap \tau G_i \tau^{-1} = \bigcap_{V \in \mathcal{V}} (V \cap \tau G_i \tau^{-1}).$$

Since every open subgroup of $G_{\hat{\mathcal{C}}}$ containing $\bar{H}$ is of the form $\bar{V}$ for some $V \in \mathcal{V}$, we have that $\bar{H} = \bigcap_{V \in \mathcal{V}} \bar{V}$.

We claim that

$$\bigcap_{V \in \mathcal{V}} \overline{V \cap \tau G_i \tau^{-1}} = \overline{H \cap \tau G_i \tau^{-1}}.$$

To see this it suffices to show that for any open subgroup W of $\tau G_i \tau^{-1}$ containing $H \cap \tau G_i \tau^{-1}$, there exists some $V \in \mathcal{V}$ such that $V \cap \tau G_i \tau^{-1} \leq W$ (indeed, since any open subgroup of $\tau \bar{G}_i \tau^{-1}$ containing $\overline{H \cap \tau G_i \tau^{-1}}$ is of the form $\overline{W}$, this would mean that every open subgroup of $\tau \bar{G}_i \tau^{-1}$ containing $\overline{H \cap \tau G_i \tau^{-1}}$ contains also some $\overline{V \cap \tau G_i \tau^{-1}}$). Choose $U \in \mathcal{V}$ satisfying the statement of Corollary 3.6 with respect to H:

$$U = U \cap \tau G_i \tau^{-1} * \cdots$$

$$H = H \cap \tau G_i \tau^{-1} * \cdots.$$

Consider the natural epimorphism of U onto $U \cap \tau G_i \tau^{-1}$. Let V be the preimage of $W \cap U \cap \tau G_i \tau^{-1}$. Then V is open and contains H, i.e., $V \in \mathcal{V}$; moreover $V \cap \tau G_i \tau^{-1} = W \cap U \cap \tau G_i \tau^{-1} = W \cap U \leq W$.

Now,

$$\bar{H} \cap \tau \bar{G}_i \tau^{-1} = \bigcap_{V \in \mathcal{V}} (\bar{V} \cap \tau \bar{G}_i \tau^{-1}) = \bigcap_{V \in \mathcal{V}} \overline{V \cap \tau G_i \tau^{-1}} = \overline{H \cap \tau G_i \tau^{-1}},$$

as desired. □

Lemma 3.11 Let $G_1, \ldots, G_m$ be groups and let H be a finitely generated closed subgroup of the free product $G = G_1 * \cdots * G_m$ (endowed with the pro-$\mathcal{C}$ topology). Fix $i \in \{1, \ldots, m\}$

and assume that the group G_i is 2-subgroup separable. Then HK and KH are closed subsets of G for any closed subgroup K of G_i.

Proof: We prove that HK is closed; for KH the proof is similar. We must show that $G \cap \bar{H}\bar{K} = HK$.

Let $S(G)$ be the standard tree of the free product $G = G_1 * \cdots * G_m$ and let D be a minimal H-invariant subtree of $S(G)$ containing v_0. Then $H\backslash D$ is finite. Choose a connected transversal Σ_H of $H\backslash D$ in D. Then by Lemma 3.5 there exists an open subgroup U of G containing H and a connected transversal Σ_U of $U\backslash S(G)$ in $S(G)$ with $\Sigma_H \subseteq \Sigma_U$ such that

$$U = \big[\ast_{w\in V(\Sigma_U)} U_w\big] * F_U = \big[\ast_{i=1}^{m}\big[\ast_{\tau\in U\backslash G/G_i} U \cap \tau G_i \tau^{-1}\big]\big] * F_U, \tag{3}$$

where F_U is the free group $\pi_1(U\backslash S(G))$, and

$$H = \big[\ast_{w\in V(\Sigma_H)} H_w\big] * F_H = \big[\ast_{i=1}^{m}\big[\ast_{\tau\in H\backslash G/G_i} H \cap \tau G_i \tau^{-1}\big]\big] * F_H,$$

where F_H is the free group $\pi_1(H\backslash D)$; moreover F_H is a free factor of F_U.

Since HK is closed if and only if $H(U \cap K)$ is closed (see the proof of Lemma 3.1) we may assume that $K \leq U$. Pick $h \in \bar{H}$ and $k \in \bar{K}$ with $hk = g \in G$. Note that $g \in U$, because $\bar{H}, \bar{K} \leq \bar{U}$ and $U = G \cap \bar{U}$, since U is open (cf. Proposition 3.2.2 in [7]). Let

$$L = (G_1)_{\hat{\mathcal{C}}} * \cdots * (G_m)_{\hat{\mathcal{C}}}$$

be the abstract free product of the completions of the groups G_i. Since $k \in \bar{G}_i$, one has $h \in \bar{H} \cap L \leq \bar{U} \cap L$. By the preceding lemma

$$\bar{H} \cap L = \big[\ast_{v\in V(\Sigma_H)} \overline{(H_v)}\big] * F_H = \big[\ast_{i=1}^{m}\big[\ast_{\tau\in H\backslash G/G_i} \overline{H \cap \tau G_i \tau^{-1}}\big]\big] * F_H$$

and

$$\bar{U} \cap L = \big[\ast_{v\in V(\Sigma_U)} \overline{(U_v)}\big] * F_U = \big[\ast_{i=1}^{m}\big[\ast_{\tau\in U\backslash G/G_i} \overline{U \cap \tau G_i \tau^{-1}}\big]\big] * F_U. \tag{4}$$

Write $h = h_{m_1} \cdots h_{m_l}$ as the reduced word of this free product decomposition of $\bar{H} \cap L$. Note that this is also a reduced word for the free product decomposition of $\bar{U} \cap L$ above. Observe that any reduced word in the free product decomposition of U above, is also reduced in the free product decomposition of $\bar{U} \cap L$.

We consider two cases. First assume that $h_{m_l} \notin \overline{U \cap G_i}$. Then, since $k \in \overline{U \cap G_i}$, we have that $g = hk = h_{m_1} \cdots h_{m_l} k$ is reduced as a word in the free product decomposition of $\bar{U} \cap L$ given above. On the other hand, $g = hk$ can be written as a product according to the free product decomposition of (3) of U; since such a product is also a product according to the free product decomposition (4) and it is unique, we deduce that the elements $h_{m_1}, \cdots, h_{m_l}, k$ are in U, and thus in G. Therefore $h, k \in G$. Finally, since H and K are closed, we deduce that $G \cap \bar{H} = H$ and $G \cap \bar{K} = K$; so, $h \in H$ and $k \in K$, in particular, $hk \in HK$. Thus, $G \cap \bar{H}\bar{K} = HK$.

Assume next that $h_{m_l} \in \overline{U \cap G_i}$. If $h_{m_l} = k^{-1}$, then $hk \in G \cap \bar{H} = H$, and we are done. Otherwise, $h_{m_l} \neq k^{-1}$, and so

$$h_{m_1} \cdots h_{m_{l-1}}(h_{m_l}k)$$

is a reduced expression for $g = hk$ in the free product (4). Again, since $g \in U$, this coincides with the unique expression for g in the free product (3). Hence,

$$h_{m_1}, \cdots, h_{m_{l-1}}, (h_{m_l}k) \in U \leq G.$$

Therefore, $h_{m_1}, \cdots, h_{m_{l-1}} \in G \cap \bar{H} = H$ and $h_{m_l}k \in G \cap \overline{U \cap G_i} = U \cap G_i$. Now, since $U \cap G_i$ is 2-separable (see Lemma 3.1), there are $h' \in H \cap G_i$, $k' \in K$ with $h'k' = h_{m_l}k$. Hence

$$g = hk = h_{m_1} \cdots h_{m_{l-1}}(h'k') = (h_{m_1} \cdots h_{m_{l-1}}h')k' \in HK,$$

as desired. □

Corollary 3.12 Let $G = G_1 * \cdots * G_m$ be a free product of groups G_i and assume G is endowed with the pro-$\mathcal{C}$ topology. Let K be a closed subgroup of G and let K_i be a closed subgroup of G_i ($i = 1, \ldots, m$) such that $K = K_1 * \cdots * K_m$. Let $S(G_{\hat{\mathcal{C}}})$, $S(\bar{K})$, $S(K)$ and $S(G)$ be the profinite graphs associated with the free pro-$\mathcal{C}$ products $G_{\mathcal{C}} = (G_1)_{\hat{\mathcal{C}}} \amalg \cdots \amalg (G_m)_{\hat{\mathcal{C}}}$, $\bar{K} = \bar{K}_1 \amalg \cdots \amalg \bar{K}_m$, $K = K_1 * \cdots * K_m$ and $G = G_1 * \cdots * G_m$, respectively. Then $S(\bar{K})$, $S(K)$ and $S(G)$ are naturally embedded in $S(G_{\hat{\mathcal{C}}})$ and

$$S(\bar{K}) \cap S(G) = S(K).$$

Proof: The embeddings are easy to check (in the case of $S(K)$ it follows from the assumption that K and K_i are closed in G and G_i, respectively, $i = 1, \ldots, m$). We need to check that if $x \in S(\bar{K}) \cap S(G)$, then $x \in S(K)$. Recall that the graph $S(G)$ consists of the G-translates of the finite graph T_m (see Section 2) with the proviso that in $S(G)$ the stabilizer of xv_i is xG_ix^{-1} where $G_0 = 1$ and the edge stabilizers are trivial (and analogously for $S(K), S(G_{\hat{\mathcal{C}}}), S(\bar{K})$). If x is an edge or a translate of v_0, then clearly $x \in S(K)$ because the stabilizer of x is trivial (e.g., if x has the form $ge_i = \tilde{k}e_1$, with $g \in G$, $\tilde{k} \in \bar{K}$, then $g = \tilde{k} \in G \cap \bar{K} = K$). Assume next that x is a translate of v_i, where $i \geq 0$. Then x has the form $gv_i = \tilde{k}v_i$; and this implies that $\tilde{k}^{-1}g \in (G_i)_{\hat{\mathcal{C}}}$ ($g \in G, \tilde{k} \in \bar{K}$), i.e., $g = \tilde{k}\tilde{g}_i$, with $\tilde{g}_i \in (G_i)_{\hat{\mathcal{C}}}$. By Lemma 3.11, we have that $g = kg_i$, with $k \in K$ and $g_i \in G_i$. Therefore $gv_i = kv_i \in S(K)$, as needed. □

Theorem 3.13 Let $G_1, \ldots, G_m$ be groups. Assume that in each G_i the product of any two finitely generated closed subgroups in the pro-$\mathcal{C}$ topology is a closed subset (i.e., each G_i is 2-product subgroup separable). Then their free product

$$G = G_1 * \cdots * G_m$$

is 2-product subgroup separable in the pro-$\mathcal{C}$ topology of G.

Proof: Let H and K be finitely generated subgroups of G which are closed in the pro-$\mathcal{C}$ topology of G. We must show that the set HK is closed in the pro-$\mathcal{C}$ topology of G. By Lemma 3.1 and Corollary 3.6 we may assume that K has the form

$$K = K_1 * \cdots * K_m,$$

where K_i is a closed subgroup of G_i ($i = 1, \dots, m$).

Let $\bar{H}$ and $\bar{K}$ denote the closures of H and K, respectively, in the pro-$\mathcal{C}$ completion $G_{\hat{\mathcal{C}}}$ of G. Note that $\overline{HK} = \bar{H}\bar{K}$. To show that HK is closed is equivalent to showing that

$$HK = (\bar{H}\bar{K}) \cap G.$$

Obviously $HK \subseteq (\bar{H}\bar{K}) \cap G$. To prove the opposite containment, let $\tilde{h} \in \bar{H}$ and $\tilde{k} \in \bar{K}$ and assume that

$$g = \tilde{h}\tilde{k} \in (\bar{H}\bar{K}) \cap G.$$

We have to show that

$$g \in HK.$$

Consider the standard trees $S(G)$ and $S(K)$ associated with the abstract free product decompositions

$$G = G_1 * \cdots * G_m \quad \text{and} \quad K = K_1 * \cdots * K_m,$$

respectively. Observe that $\bar{K} = \bar{K}_1 \amalg \cdots \amalg \bar{K}_m$, where $\bar{K}_i$ is the closure of K_i in $G_{\hat{\mathcal{C}}}$ (cf. [7], Corollary 9.1.7). Consider the standard pro-$\mathcal{C}$ trees $S(G_{\hat{\mathcal{C}}})$ and $S(\bar{K})$ associated with the free pro-$\mathcal{C}$ product decompositions

$$G_{\hat{\mathcal{C}}} = (G_1)_{\hat{\mathcal{C}}} \amalg \cdots \amalg (G_m)_{\hat{\mathcal{C}}} \quad \text{and} \quad \bar{K} = \bar{K}_1 \amalg \cdots \amalg \bar{K}_m,$$

respectively.

Since K_i is closed in G_i (and thus in G) for each i, the canonical map of graphs $S(K) \longrightarrow S(\bar{K})$ is an embedding. We shall think of $S(G)$ as being canonically embedded in $S(G_{\hat{\mathcal{C}}})$, of $S(K)$ as being canonically embedded in $S(\bar{K})$ and in $S(G)$, and of $S(\bar{K})$ as being canonically embedded in $S(G_{\hat{\mathcal{C}}})$. Thus we have the following diagram of trees (abstract and profinite):

$$\begin{array}{ccc} S(K) & \hookrightarrow & S(G) \\ \downarrow & & \downarrow \\ S(\bar{K}) & \hookrightarrow & S(G_{\hat{\mathcal{C}}}) \end{array}$$

Remark that all the quotient graphs $G_{\hat{\mathcal{C}}}\backslash S(G_{\hat{\mathcal{C}}})$, $\bar{K}\backslash S(\bar{K})$, $G\backslash S(G)$ and $K\backslash S(K)$ are isomorphic to the finite tree T_m introduced in Section 2; as we explained there, we shall identify T_m with its canonical transversal in $S(K)$; in particular, $v_0 = 1K_0$, where K_0 is the trivial group.

Since $g \in G$, it can be written as a finite product of elements from $G_1, \ldots, G_m$; hence the geodesic $[v_0, gv_0]$ is finite, and therefore so is

$$\tilde{h}^{-1}[v_0, gv_0] = [\tilde{h}^{-1}v_0, \tilde{k}v_0].$$

Let

$$D = \bigcup_{j \in J} H[v_0, r_j v_0],$$

where $\{r_j \mid j \in J\}$ is a finite set of generators for H; then D is the minimal H-invariant subtree of $S(G)$ containing v_0. Consider the closure

$$\bar{D} = \bigcup_{j \in J} \bar{H}[v_0, r_j v_0]$$

of D in $S(G_{\hat{\mathcal{C}}})$. Note that we have equal finite quotient graphs

$$\bar{H} \backslash \bar{D} = H \backslash D$$

by Lemma 3.8. Observe that $\bar{D}$ is a pro-$\mathcal{C}$ tree. It follows that

$$[\tilde{h}^{-1}v_0, \tilde{k}v_0] \subseteq \bar{D} \cup S(\bar{K}).$$

If $\tilde{h} \in \bar{K}$, then

$$\tilde{h}\tilde{k} \in \bar{K} \cap G = K,$$

since K is closed, and thus the result follows. Hence we may assume that $\tilde{h} \notin \bar{K}$. Now, since $[\tilde{h}^{-1}v_0, \tilde{k}v_0]$ is finite, there exists a vertex

$$v' \in [\tilde{h}^{-1}v_0, \tilde{k}v_0] \cap S(\bar{K})$$

such that $[\tilde{h}^{-1}v_0, v']$ is minimal.

We claim that $v' \in [\tilde{h}^{-1}v_0, v_0]$. Indeed, otherwise (since $[\tilde{h}^{-1}v_0, v']$ is finite) there exists a vertex

$$w \in [\tilde{h}^{-1}v_0, v']$$

such that $w \in [\tilde{h}^{-1}v_0, v_0]$ but none of the edges of $[w, v']$ is in $[\tilde{h}^{-1}v_0, v_0]$. Then

$$[w, v'] \cap ([w, v_0] \cup S(\bar{K}))$$

is a finite tree (since the intersection is nonempty) consisting of the two vertices w and v' but no edges, a contradiction. This proves the claim. In particular $v' \in \bar{D}$. Therefore, one has

$$[\tilde{h}^{-1}v_0, v'] \subseteq [\tilde{h}^{-1}v_0, v_0] \cap [\tilde{h}^{-1}v_0, \tilde{k}v_0].$$

Clearly $[v', \tilde{k}v_0]$ is a finite path in $S(\bar{K})$. Hence $[\tilde{k}^{-1}v', v_0]$ is finite. On the other hand,

$$[\tilde{k}^{-1}v', v_0] = \tilde{k}^{-1}[v', \tilde{k}v_0] \subseteq \tilde{k}^{-1}[\tilde{h}^{-1}v_0, \tilde{k}v_0] = [\tilde{k}^{-1}\tilde{h}^{-1}v_0, v_0] = [g^{-1}v_0, v_0] \subseteq S(G),$$

and so $\tilde{k}^{-1}v' \in S(G) \cap S(\bar{K}) = S(K)$ (see Corollary 3.12). Then, there exists $k \in K$ such that $kv_i = \tilde{k}^{-1}v'$, for some $i = 0, \ldots, n$. This means that $v' = \tilde{k}kv_i$. Now

$$\tilde{h}\tilde{k} \in G \quad \text{if and only if} \quad \tilde{h}\tilde{k}k \in G;$$

and

$$\tilde{h}\tilde{k} \in HK \quad \text{if and only if} \quad \tilde{h}\tilde{k}k \in HK.$$

Hence, replacing $\tilde{k}k$ for $\tilde{k}$, we may assume that $v' = \tilde{k}v_i$, for some $i = 0, \ldots, n$.

Denote by

$$\varphi : \bar{D} \longrightarrow \bar{H}\backslash\bar{D} = H\backslash D$$

the canonical morphism of graphs. Observe that

$$T = \bar{D} \cap S(\bar{K})$$

is a pro-$\mathcal{C}$ subtree of $S(G_{\hat{\mathcal{C}}})$. We shall prove first that the quotient graph $(\bar{H} \cap \bar{K})\backslash T$ is finite. To see this consider the natural action of $\bar{H} \cap \bar{K}$ on the space

$$T' = T \cap ((G_{\hat{\mathcal{C}}})v_0 \cup E(S(G_{\hat{\mathcal{C}}})).$$

We prove first that the set $(\bar{H} \cap \bar{K})\backslash T'$ has the same cardinality as $\varphi(T')$, and so it is finite. Indeed, note that φ induces a surjection of sets

$$\bar{\varphi} : (\bar{H} \cap \bar{K})\backslash T' \longrightarrow \varphi(T').$$

Now, suppose $t, t' \in T'$ and $xt = t'$ for some $x \in \bar{H}$; in particular t and t' are in the same $G_{\hat{\mathcal{C}}}$-orbit. Since $t, t' \in S(\bar{K})$, there exists $\tilde{k}' \in \bar{K}$ such $\tilde{k}'t = t'$. So,

$$x^{-1}\tilde{k}'t = t.$$

Since $t \in T'$, its stabilizer is trivial. Therefore,

$$x = \tilde{k}' \in \bar{H} \cap \bar{K}.$$

Thus, $\bar{\varphi}$ is a bijection.

Since the edges of T are in T', it follows that $(\bar{H} \cap \bar{K})\backslash T$ has only finitely many edges, and so it is a finite graph. Let

$$\rho : T \longrightarrow (\bar{H} \cap \bar{K})\backslash T$$

be the canonical epimorphism of graphs. Then we have a commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{\rho} & (\bar{H} \cap \bar{K})\backslash T \\ \big\downarrow & & \big\downarrow \psi \\ \bar{D} & \xrightarrow{\varphi} & \bar{H}\backslash\bar{D} = H\backslash D \end{array}$$

where the restriction of ψ to $(\bar{H}\cap\bar{K})\backslash T'$ (and in particular, to the set of edges of $(\bar{H}\cap\bar{K})\backslash T$) is an injection.

We claim that there exists a connected transversal Σ of ρ containing v_0 such that $\Sigma \subseteq D \subseteq S(G)$. Clearly $\rho(v_0)$ lifts to $v_0 \in D$. Let Δ be a maximal subgraph of $\rho(T)$ for which there is a ρ-transversal Σ which is in D such that $v_0 \in \Sigma$. Remark that

$$\Sigma \subseteq D \cap S(\bar{K}) \subseteq S(G) \cap S(\bar{K}) = S(K)$$

(the last equality follows from Corollary 3.12). If $\rho(\Sigma) \neq \rho(T)$ then there exists a vertex w of Σ such that $\rho(w)$ has an incident edge $\bar{e} \in \rho(T)$ which is not in $\rho(\Sigma)$. Let e be an edge of T incident with w such that $\rho(e) = \bar{e}$. Say $w = yv_i$ for some i $(0 \leq i \leq m)$ and some $y \in G$. Since $\bar{H}\backslash\bar{D} = H\backslash D$ (see Lemma 3.8), there is an edge e' of D incident with w such that $\varphi(e') = \bar{e}$. Note that the stabilizer of w in $S(G_{\hat{\mathcal{C}}})$ is $(G_i)^y_{\hat{\mathcal{C}}} = y(G_i)_{\hat{\mathcal{C}}}y^{-1}$. Hence, since $e, e' \in \bar{D}$, there exists $\hat{h} \in \bar{H} \cap (G_i)^y_{\hat{\mathcal{C}}}$ with $\hat{h}e = e'$. If $i = 0$, then $G_0 = 1$; so $\hat{h} = 1$; therefore $e = e'$ is in T and in D. This would contradict the maximality of Δ. Thus we may assume that $1 \leq i \leq m$. By Lemma 3.10 we have that $\bar{H} \cap (G_i)^g_{\hat{\mathcal{C}}} = \overline{H \cap G^g_i}$. Let v_e and $v_{e'}$ be the vertices different from w of e and e', respectively. Then $\hat{h}v_e = v_{e'}$. On the other hand $v_e = \hat{k}v_0$ for some $\hat{k} \in \bar{K}$ and $v_{e'} = y'v_0$ for some $y' \in G$, since $v_e \in S(\bar{K})$ and $v_{e'} \in D \subseteq S(G)$. Therefore, $\hat{h}\hat{k} = y'$. By Lemma 3.11 $\hat{h}\hat{k} = hk$, for some $h \in H \cap G^y_i$, $k \in K$. It follows that

$$h^{-1}v_{e'} = ky'^{-1}v_{e'} = kv_0 \in S(K).$$

Since $h^{-1}w = w$, we deduce that $h^{-1}e' \in S(K)$ and $h^{-1}e'$ is incident with w; hence $h^{-1}e' \in D \cap S(K) \subseteq T$. Since $\rho(h^{-1}e') = \bar{e}$, we get a contradiction to the maximality of Δ. This proves the claim.

As pointed out above, $\Sigma \subseteq S(K)$. Now, since $v' \in \bar{D} \cap S(\bar{K}) = T$, there exists some $\alpha \in \bar{H} \cap \bar{K}$ such that $\alpha v' \in \Sigma$; hence $\alpha\tilde{k}v_i \in \Sigma \subseteq S(K)$. Therefore, $\alpha\tilde{k}v_i = xv_i$, for some $x \in K$. Since the stabilizer of v_i in $\bar{K}$ is $\bar{K}_i$, we deduce that $\alpha\tilde{k} \in K\bar{K}_i$.

Write $\alpha\tilde{k} = k\hat{k}_i$, where $k \in K, \hat{k}_i \in \bar{K}_i$. Note that

$$\tilde{h}\alpha^{-1}\alpha\tilde{k} = \tilde{h}\tilde{k} = g \in G.$$

So,

$$(\tilde{h}\alpha^{-1})k\hat{k}_i = g \in G.$$

It follows that

$$(k^{-1}(\tilde{h}\alpha^{-1})k)\hat{k}_i = k^{-1}g \in G.$$

By Lemma 3.11 there exist $h \in H, k_i \in K_i$ such that $(k^{-1}hk)k_i = k^{-1}g \in G$ and so $h(kk_i) = g$, as required. □

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