Free-by-Demushkin pro-p groups

Dessislava H. Kochloukova *,
IMECC-UNICAMP,Cx. P. 6065,
13083-970 Campinas, SP, Brazil,
Pavel Zalesskii *
Department of Mathematics, University of Brasília
70910-900 Brasília DF, Brazil,
desi@ime.unicamp.br, pz@mat.unb.br

Abstract

We give an example of a short exact sequence $1 \to N \to G \to D \to 1$ of pro-p groups such that the cohomological dimension $\mathrm{cd}(G) = 2$, G is (topologically) finitely generated, N is a free pro-p group of infinite rank, D is a Demushkin group, for every closed subgroup S of G containing N and any natural number n the inflation map $\mathrm{H}^2(S/N, \mathbb{Z}/(p^n)) \to \mathrm{H}^2(S, \mathbb{Z}/(p^n))$ is an isomorphism but G is not a free pro-p product of a free pro-p group by a Demushkin group. This is a group theoretic version of a question raised by T. Würfel for some special Galois groups.

1 Introduction

In [13] Würfel proved the following

Theorem 1. *[13] Let F be a field with separable closure F_s and absolute Galois group $G = \mathrm{Gal}(F_s/F)$. Suppose G is a finitely generated one-relator pro-p group where the prime p is different from* $\mathrm{char}(F)$ *and F contains*

*Both authors are partially supported by "bolsa de produtividade de pesquisa" from CNPq, Brazil and CNPq grant 470272/2003-1

all p-power roots of unity. Then there is a normal closed free pro-p subgroup N of G such that G/N is a Demushkin group and the inflation map $\mathrm{H}^2(S/N, \mathbb{Z}/(p^n)) \to \mathrm{H}^2(S, \mathbb{Z}/(p^n))$ is an isomorphism for every closed subgroup S of G containing N, and all integers n.

In the same paper he asked whether the condition in this theorem implies that G is free pro-p product of a Demushkin group and a free pro-p group.

In this paper we answer the group theoretic version of Würfel's question negatively by the means of the following example.

Theorem 2. *Let G be the pro-p group with three (topological) generators x, y, z and one defining relation $z^{p^s} = [x, y]$ where $s \geq 1$ if $p \neq 2$ and $s \geq 2$ for $p = 2$. Let N be the normal closed subgroup of G generated by z and define $D = G/N$. Then*

a) $\mathrm{cd}(G) = 2$;

b) D is the Demushkin group $\mathbb{Z}_p \times \mathbb{Z}_p$;

c) N is a free pro-p group of infinite rank;

d) For every closed subgroup S of G containing N the inflation map $\mathrm{H}^2(S/N, \mathbb{F}_p) \to \mathrm{H}^2(S, \mathbb{F}_p)$ is an isomorphism;

e) For every closed subgroup S of G containing N and any natural number n the inflation map $\mathrm{H}^2(S/N, \mathbb{Z}/(p^n)) \to \mathrm{H}^2(S, \mathbb{Z}/(p^n))$ is an isomorphism;

f) G is not a free pro-p product of a free pro-p group with a Demushkin group.

We observe that the class of groups considered in Theorem 2 cannot be realised as Galois groups in the sense of Würfel's question as such groups would be Galois groups of maximal p-extensions of fields and by [7, Thm. 1.2] for such Galois groups with 3 (topological) generators the second cohomology with coeficients in $\mathbb{F}_p$ has dimension 3 over $\mathbb{F}_p$ and therefore cannot be 1 relator. In fact, later [14, Remark, p. 210] Würfel observed that the answer to his question is affirmative if the natural epimorphism $G \to G/N$ splits. We do not know whether field theory inforces that the homomorphism $G \to G/N$ splits.

Finally we want to express our gratitute to Prof. Dr. Antonio Engler for sugesting and discussing the question, providing and explaining the reference [7] to us and the encouragement along the way.

2 Some preliminary results

Demushkin groups D are one relator pro-p groups of cohomological dimension 2 with the property that the cup product

$$\cup : \mathrm{H}^1(D,\mathbb{F}_p) \times \mathrm{H}^1(D,\mathbb{F}_p) \to \mathrm{H}^2(D,\mathbb{F}_p) \simeq \mathbb{F}_p$$

is a non-singular bilinear form. There are two invariants associated to a Demushkin group: the minimal number of (topological) generators d and q that is either ∞ or a power of the prime p. We remind the reader several important properties of Demushkin groups. The case of $q \neq 2$ is done in [3], [4]. Another excellent reference for this case is [12, 12.3.1, 12.3.6]

Theorem 3. *[3], [4] Let D be a Demushkin group with invariants d, q and suppose that $q \neq 2$. Then d is even and D is isomorphic to F/R, where F is a free pro-p group with basis $x_1, \dots, x_d$ and R is generated as a normal closed subgroup by*

$$x_1^q[x_1, x_2] \cdots [x_{d-1}, x_d]$$

where for $q = \infty$ we define $x_1^\infty = 1$. Furthermore all groups having such presentations are Demushkin.

In the case when D is a Demushkin group with $q = 2$ the classification was completed by J.-P. Serre [11] and J. Labute [8].

Theorem 4. *[11] Let D be a Demushkin pro-2 group with invariants d, q and suppose that $q = 2$ and d is odd. Then D is isomorphic to F/R, where F is a free pro-2 group with basis $x_1, \dots, x_d$ and R is generated as a normal closed subgroup by*

$$x_1^2 x_2^{2^f}[x_2, x_3] \cdots [x_{d-1}, x_d]$$

for some integer $f \geq 2$ or ∞. Furthermore all groups having such presentations are Demushkin.

Theorem 5. *[8] Let D be a Demushkin pro-2 group with d even and $q = 2$. Then D is isomorphic to F/R, where F is a free pro-2 group with basis $x_1, \dots, x_d$ and R is generated as a normal closed subgroup either by*

$$x_1^{2^f+2}[x_1, x_2][x_3, x_4] \cdots [x_{d-1}, x_d] \text{ for some integer } f \geq 2 \text{ or } \infty,$$

or by

$$x_1^2[x_1, x_2]x_3^{2^f}[x_3, x_4] \cdots [x_{d-1}, x_d] \text{ for some integer } f \geq 2 \text{ or } \infty, d \geq 4.$$

Furthermore all groups having such presentations are Demushkin.

3 Some properties of the group G from Theorem 2

In this section G is the pro-p group from Theorem 2. We denote by $\mathbb{Z}_p[[G]]$ the completed group algebra of G with coefficients in $\mathbb{Z}_p$. Though discrete groups with one defining relation that is not a proper power are always of cohomological dimension ≤ 2 [1] one related pro-p groups with one defining relation that is not a p-th power are not automatically of cohomological dimension ≤ 2 [6]. Thus part a) from Theorem 2 cannot be deduced directly from the fact that the group G is a 1-relator, pro-p torsion-free group.

Lemma 1. *The pro-p group G has cohomological dimension 2.*

Proof. Note that G is a not a free pro-p group as the relator $z^{p^s}[x,y]^{-1}$ is in the Frattini subgroup of the free pro-p group with a basis x, y, z, hence by [10, Cor. 7.5.2] $\mathrm{cd}(G) \neq 1$. Obviously, G is the free amalgamated pro-p product $C *_H F$, where $C = \langle z \rangle \simeq \mathbb{Z}_p$, F the free pro-p group with basis x, y, $H = \langle t \rangle \simeq \mathbb{Z}_p$, and the embeddings $H \to C$ and $H \to F$ are given by $t \to z^{p^s}$ and $t \to [x,y]$, respectively. By [10, Exer. 9.2.6(b)] this free pro-p amalgamated product is proper. Hence by [10, Prop. 9.2.13(a)] $\mathrm{cd}(G) \leq max\{\mathrm{cd}(C), \mathrm{cd}(F), \mathrm{cd}(H) + 1\} = 2$. □

Lemma 2. *Let $F = F(x,y)$ be a free pro-p group with basis x, y and V be an open subgroup of $F(x,y)$ of index p. Then there exists a basis w_1, w_2 of F such that $[x,y] = [w_1,w_2]$ and V is (topologically) generated by $w_1^p, w_2, w_2^{w_1}, \ldots, w_2^{w_1^{p-1}}$.*

Proof. Let $\theta : F \to \mathbb{F}_p$ be a homomorphism of pro-p groups with kernel V, $\theta(x) = \beta$ and $\theta(y) = \alpha$ where $\mathbb{F}_p$ is the field with p elements. First assume that $\alpha \neq 0$. We use the commutator calculations

$$[ab, c] = [a,c]^b \cdot [b,c], \quad [a, bc] = [a,c] \cdot [a,b]^c, \text{ where } [a,b] = a^{-1}b^{-1}ab.$$

Define

$$y_1 = y^{n_1}x, y_2 = y \text{ and } w_1 = y_1, w_2 = y_1^{n_2}y_2,$$

for some $n_1, n_2 \in \mathbb{Z}$. Then both pairs $\{y_1, y_2\}$ and $\{w_1, w_2\}$ are bases of F. We first prove that $w_1^p, w_2 \in V$ for some choice of n_1, n_2. Using the above commutator calculations we get

$$[y_1, y_2] = [y^{n_1}x, y] = [y^{n_1}, y]^x \cdot [x,y] = [x,y],$$

$$[w_1, w_2] = [y_1, y_1^{n_2} y_2] = [y_1, y_2] \cdot [y_1, y_1^{n_2}]^{y_2} = [y_1, y_2].$$

Finally $\theta(w_1) = \bar{n}_1\alpha + \beta$ and $\theta(w_2) = (\bar{n}_1\alpha + \beta)\bar{n}_2 + \alpha$ in $\mathbb{F}_p$, where $\bar{n}_i$ is the image of n_i in $\mathbb{F}_p$. Thus it is sufficient to solve in $\mathbb{F}_p$ the system for $\bar{n}_1, \bar{n}_2$: $\bar{n}_1\alpha + \beta = 1, \bar{n}_2 + \alpha = 0$. Then $\theta(w_1) = 1$ and $\theta(w_2) = 0$.

If $\alpha = 0$ we have $\beta \neq 0$ and can define $w_1 = x, w_2 = y$. Then $[x, y] = [w_1, w_2]$, $\theta(w_1) \neq 0$ and $\theta(w_2) = 0$.

In both cases the closed normal subgroup W of F generated by w_1^p and w_2 is of index p in F and is contained in V, hence $V = W$. Therefore V is (topologically) generated by $w_1^p, w_2, w_2^{w_1}, \ldots, w_2^{w_1^{p-1}}$.

□

From now on for a set A we denote by $F(A)$ the free pro-p group with basis A.

Lemma 3. *Let $\{z_1, \ldots, z_n\}$ and $\{x, y\}$ be disjoint sets. Let*

$$H = F(z_1, \ldots, z_n) *_{z_1^{p^s} \cdots z_n^{p^s} = [x,y]} F(x, y)$$

be the free amalgamated pro-p product and H_0 be the normal closed subgroup of H generated by $z_1, \ldots, z_n$. Then every open subgroup U of H of index p such that $z_1, \ldots, z_n \in U$ has a similar presentation i.e. $U \simeq F(\tilde{z}_1, \ldots, \tilde{z}_k) *_{\tilde{z}_1^{p^s} \cdots \tilde{z}_k^{p^s} = [\tilde{x}, \tilde{y}]} F(\tilde{x}, \tilde{y})$ and H_0 is the normal closed subgroup of U generated by $\tilde{z}_1, \ldots, \tilde{z}_k$. Furthermore as sets

$$\{\tilde{z}_1, \ldots, \tilde{z}_k\} = \{z_1, z_1^{w_1}, \ldots, z_1^{w_1^{p-1}}, z_2, z_2^{w_1}, \ldots, z_2^{w_1^{p-1}}, \ldots, z_n, z_n^{w_1}, \ldots, z_n^{w_1^{p-1}}\}$$

and $\tilde{x} = w_1^p, \tilde{y} = w_2$ for some basis w_1, w_2 of $F(x, y)$.

Proof. By Lemma 2 there exists a basis w_1, w_2 of $F(x, y)$ such that $[x, y] = [w_1, w_2]$ and $U \cap F(x, y)$ is the normal closed subgroup of $F(x, y)$ generated by w_1^p and w_2. Note that the open subgroups of H containing $z_1, \ldots, z_n$ correspond to the open subgroups of $F(x, y)$ containing $[x, y]$. Then changing $\{x, y\}$ to $\{w_1, w_2\}$ we can assume that U is the normal closed subgroup of H generated by $z_1, \ldots, z_n, x^p, y$. By the Reidemeister-Schreier method [2, Ch. 7,Thm. 7] we get a generating set and a set of relations for U. As a generating set $\mathcal{X}$ we have

$$\{z_1, z_1^x, \ldots, z_1^{x^{p-1}}, z_2, z_2^x, \ldots, z_2^{x^{p-1}}, \ldots, z_n, z_n^x, \ldots, z_n^{x^{p-1}}, y, y^x, \ldots, y^{x^{p-1}}, x^p\}$$

and relations that are conjugates of the relation of H by the representatives $\{1, x, x^2, \ldots, x^{p-1}\}$ of the left cosets of U in H

$$z_1^{p^s} z_2^{p^s} \cdots z_n^{p^s} = [x, y] = (y^x)^{-1} y,$$

$$(z_1^x)^{p^s} (z_2^x)^{p^s} \cdots (z_n^x)^{p^s} = (y^{x^2})^{-1} y^x,$$

$$(z_1^{x^2})^{p^s} (z_2^{x^2})^{p^s} \cdots (z_n^{x^2})^{p^s} = (y^{x^3})^{-1} y^{x^2},$$

$$\cdots$$

$$(z_1^{x^{p-1}})^{p^s} (z_2^{x^{p-1}})^{p^s} \cdots (z_n^{x^{p-1}})^{p^s} = (y^{x^p})^{-1} y^{x^{p-1}}.$$

We use the first $p-1$ relations to eliminate the elements

$$\mathcal{T} = \{y^x, y^{x^2}, \ldots, y^{x^{p-1}}\}$$

from the generating set $\mathcal{X}$. We multiply the relations left to right starting with the last one and going backwards and most of the terms in the right hand side cancel to get a new relation r_1 of U. We get

$$(z_1^{x^{p-1}})^{p^s} (z_2^{x^{p-1}})^{p^s} \cdots (z_n^{x^{p-1}})^{p^s} \cdots (z_1^x)^{p^s} (z_2^x)^{p^s} \cdots (z_n^x)^{p^s} z_1^{p^s} z_2^{p^s} \cdots z_n^{p^s} =$$

$$(y^{x^p})^{-1} y = [x^p, y].$$

Thus $U \simeq F(\mathcal{A}) *_{r_1} F(y, x^p)$, where $\mathcal{A} = \mathcal{X} \setminus (\mathcal{T} \cup \{y, x^p\})$ and the relation r_1 is of the form a product of p^s-th powers of the elements of $\mathcal{A}$ in some order $= [x^p, y]$. Finally the Schreier method [2, Ch. 7,Thm. 4] implies that H_0 is the normal closed subgroup of U generated by $\tilde{z}_1, \ldots, \tilde{z}_k$. □

Proposition 1. *Let S be an open subgroup of G such that $N \subseteq S$. Then the inflation map* $\mathrm{H}^2(S/N, \mathbb{F}_p) \to \mathrm{H}^2(S, \mathbb{F}_p) \simeq \mathbb{F}_p$ *is an isomorphism.*

Proof. By definition $G = F(z) *_{z^{p^s} = [x,y]} F(x, y)$ and S is a subgroup of finite index in G containing the normal closed subgroup N of G generated by z. By repeatedly applying Lemma 3 one deduces that S is the amalgamated free pro-p product $F(z_1, \ldots, z_k) *_r F(x, y)$, where $r = [x, y](z_1^{p^s} \cdots z_k^{p^s})^{-1}$ and N is the normal closure of $F(z_1, \ldots, z_k)$ in S. As S is 1-relator group $dim_{\mathbb{F}_p} \mathrm{H}^2(S, \mathbb{F}_p) = 1$. Note that S/N is one relator pro-p group with generators x, y and one defining relation $\tilde{r} = [x, y]$. Then there is a commutative diagram with rows short exact sequences of pro-p groups

$$\begin{array}{ccccccccc} 1 & \to & K & \to & F & \to & S & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & K_1 & \to & F_1 & \to & S/N & \to & 1 \end{array}$$

where $F = F(z_1, \ldots, z_k, x, y)$, $F_1 = F(x, y)$ are free pro-p groups with K the normal closed subgroup of F generated by r and K_1 the normal closed subgroup of F_1 generated by $\tilde{r}$. The vertical maps are induced by the epimorphism $F \to F_1$ sending $z_1, \ldots, z_k$ to 1 and fixing x and y. This induces a commutative diagram

$$\begin{array}{ccccccc} 0 \leftarrow \mathrm{H}^2(S, \mathbb{F}_p) \leftarrow & \mathrm{H}^1(K, \mathbb{F}_p)^S \leftarrow & \mathrm{H}^1(F, \mathbb{F}_p) \leftarrow & \mathrm{H}^1(S, \mathbb{F}_p) \leftarrow 0 \\ \uparrow & \uparrow & \uparrow & \uparrow \\ 0 \leftarrow \mathrm{H}^2(S/N, \mathbb{F}_p) \leftarrow & \mathrm{H}^1(K_1, \mathbb{F}_p)^{S/N} \leftarrow & \mathrm{H}^1(F_1, \mathbb{F}_p) \leftarrow & \mathrm{H}^1(S/N, \mathbb{F}_p) \leftarrow 0 \end{array}$$

where the rows are the 5-term exact sequence in cohomology and the vertical maps are the inflation maps. As the maps $\mathrm{H}^1(S, \mathbb{F}_p) \to \mathrm{H}^1(F, \mathbb{F}_p)$ and $\mathrm{H}^1(S/N, \mathbb{F}_p) \to \mathrm{H}^1(F_1, \mathbb{F}_p)$ are isomorphisms, we have a commutative square with row maps isomorphisms

$$\begin{array}{ccc} \mathrm{H}^2(S, \mathbb{F}_p) & \leftarrow & \mathrm{H}^1(K, \mathbb{F}_p)^S \\ \uparrow & & \uparrow \\ \mathrm{H}^2(S/N, \mathbb{F}_p) & \leftarrow & \mathrm{H}^1(K_1, \mathbb{F}_p)^{S/N} \end{array}$$

By the proof of [10, Prop. 7.8.2] there is an isomorphism $\mathrm{Hom}(K, \mathbb{F}_p)^S = \mathrm{H}^1(K, \mathbb{F}_p)^S \longrightarrow \mathbb{F}_p$ sending f to $f(r)$ and similarly there is an isomorphism $\mathrm{Hom}(K_1, \mathbb{F}_p)^{S/N} = \mathrm{H}^1(K_1, \mathbb{F}_p)^{S/N} \longrightarrow \mathbb{F}_p$ sending g to $g(\tilde{r})$. Thus the right vertical inflation map in the above diagram is an isomorphism, hence the left vertical inflation map in the above diagram is an isomorphism. □

Proposition 2. *Let S be a closed subgroup of G of infinite index containing N. Then $\mathrm{H}^2(S, \mathbb{F}_p) = 0$ and $\mathrm{H}^2(S/N, \mathbb{F}_p) = 0$. In particular S and N are free pro-p groups.*

Proof. We think of S as the intersection of the open subgroups $\{U_\alpha\}_\alpha$ of G containing S. Thus S is the inverse limit of the inverse system $\{U_\alpha\}_\alpha$ with homomorphisms inclusions. Therefore $\mathrm{H}^2(S, \mathbb{F}_p)$ is the direct limit of $\{\mathrm{H}^2(U_\alpha, \mathbb{F}_p)\}_\alpha$ with homomorphisms that are the restriction maps $\mathrm{H}^2(U_\alpha, \mathbb{F}_p) \to \mathrm{H}^2(U_\beta, \mathbb{F}_p)$ for $U_\beta \subset U_\alpha$. We aim to show that this restriction map is always zero by showing this for the case when U_β is a subgroup of index p in U_α. Note that this will imply that $\mathrm{H}^2(S, \mathbb{F}_p) = 0$ and hence by [10, Cor. 7.1.6] $\mathrm{cd}(S) < 2$ i.e. S is a pro-p group of cohomological dimension 1. Then by [10, Thm. 7.5.1] S is a free pro-p group. In particular for $S = N$ we get that N is a free pro-p group.

Consider the commutative square for a subgroup U_β of index p in U_α, $S \subset U_\beta$,

$$\begin{array}{ccc} \mathrm{H}^2(U_\alpha, \mathbb{F}_p) & \leftarrow & \mathrm{H}^2(U_\alpha/N, \mathbb{F}_p) \\ \downarrow & & \downarrow \\ \mathrm{H}^2(U_\beta, \mathbb{F}_p) & \leftarrow & \mathrm{H}^2(U_\beta/N, \mathbb{F}_p) \end{array}$$

where the row maps are the inflation maps, hence by Proposition 1 are isomorphisms and the vertical maps are the restriction maps. Note that U_α/N is a subgroup of finite index of $G/N \simeq \mathbb{Z}_p \times \mathbb{Z}_p$, so $U_\alpha/N \simeq \mathbb{Z}_p \times \mathbb{Z}_p$ and U_β/N is the subgroup $\mathbb{Z}_p \times (p\mathbb{Z}_p)$. We claim that the right vertical map is the zero one. Indeed by [10, Lemma 7.4.1] the rows of the following commutative diagram (of finite abelian groups of exponent p) are isomorphisms

$$\begin{array}{ccc} \mathrm{H}^2(\mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{F}_p) & \simeq & \mathrm{H}^1(\mathbb{Z}_p, \mathrm{H}^1(\mathbb{Z}_p, \mathbb{F}_p)) \\ \downarrow & & \downarrow \\ \mathrm{H}^2(\mathbb{Z}_p \times (p\mathbb{Z}_p), \mathbb{F}_p) & \simeq & \mathrm{H}^1(\mathbb{Z}_p, \mathrm{H}^1(p\mathbb{Z}_p, \mathbb{F}_p)) \end{array}$$

where the horizontal isomorphisms are induced by the Lyndon-Hochschild-Serre spectral sequence for group extensions, the left vertical map is the restriction map. The right vertical map is induced by the restriction map $\mathrm{H}^1(\mathbb{Z}_p, \mathbb{F}_p) \longrightarrow \mathrm{H}^1(p\mathbb{Z}_p, \mathbb{F}_p)$ and this restriction map is zero by the natural isomorphism $\mathrm{H}^1(\mathbb{Z}_p, \mathbb{F}_p) \simeq \mathrm{Hom}(\mathbb{Z}_p, \mathbb{F}_p)$. In particular the left vertical map is zero, as claimed.

Finally we note that S/N is either the trivial group or a closed subgroup of infinite index in $G/N \simeq \mathbb{Z}_p \times \mathbb{Z}_p$, hence $S/N \simeq \mathbb{Z}_p$. In both cases $\mathrm{H}^2(S/N, \mathbb{F}_p) = 0$. □

Lemma 4. *Let $\pi : H \to M$ be an epimorphism of pro-p groups such that the inflation map $\mathrm{H}^2(M, \mathbb{F}_p) \to \mathrm{H}^2(H, \mathbb{F}_p)$ is an isomorphism. Then for every natural number $n \geq 1$ the inflation map $\mathrm{H}^2(M, \mathbb{Z}/(p^n)) \to \mathrm{H}^2(H, \mathbb{Z}/(p^n))$ is an isomorphism.*

Proof. We use induction on n. We assume the lemma holds for some fixed $n \geq 1$. The short exact sequence $0 \to \mathbb{Z}/(p) \to \mathbb{Z}/(p^{n+1}) \to \mathbb{Z}/(p^n) \to 0$ yields a diagram with two long exact sequences in cohomology in which the vertical maps are the inflation maps

$$\begin{array}{ccccc} \mathrm{H}^2(H, \mathbb{Z}/(p)) & \to & \mathrm{H}^2(H, \mathbb{Z}/(p^{n+1})) & \to & \mathrm{H}^2(H, \mathbb{Z}/(p^n)) \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{H}^2(M, \mathbb{Z}/(p)) & \to & \mathrm{H}^2(M, \mathbb{Z}/(p^{n+1})) & \to & \mathrm{H}^2(M, \mathbb{Z}/(p^n)) \end{array}$$

As the leftmost and the rightmost vertical maps are isomorphisms, the middle one is an isomorphism. □

4 Proof of Theorem 2

We complete the proof of Theorem 2 using the results from the previous sections.

a) $\mathrm{cd}(G) = 2$ is done in Lemma 1.

b) $D = \mathbb{Z}_p \times \mathbb{Z}_p$ is a Demushkin group by Theorem 3.

c) N is a free pro-p group by Proposition 2. If the rank of N is finite then by the main result of [5] the quotient group $G/N \simeq D$ has virtually finite cohomological dimension $\mathrm{cd}(G) - \mathrm{cd}(N) = 1$. But D is a group of cohomological dimension 2, a contradiction.

d) For every closed subgroup S of G containing N the inflation map $\mathrm{H}^2(S/N, \mathbb{F}_p) \to \mathrm{H}^2(S, \mathbb{F}_p)$ is an isomorphism by Proposition 1 and Proposition 2.

e) By d) and Lemma 4 for every closed subgroup S of G containing N and any natural number n the inflation map $\mathrm{H}^2(S/N, \mathbb{Z}/(p^n)) \to \mathrm{H}^2(S, \mathbb{Z}/(p^n))$ is an isomorphism.

f) Suppose G is a free pro-p product of a free pro-p group M_1 with a Demushkin group D_1. Then the minimal number of generators of G is the sum of the minimal number of generators of M_1 and D_1. We remind the reader that the minimal number of generators of G is 3.

1) Assume now that the invariant q of D_1 is not 2. By Theorem 3 a Demushkin group with invariant $q \neq 2$ has even number of generators, hence M_1 is $\mathbb{Z}_p$ and D_1 is two-generated. There are two options for D_1. If D_1 is $\mathbb{Z}_p \times \mathbb{Z}_p$ then G has a pro-p presentation $\langle y_1, y_2, y_3 \mid [y_1, y_2] = 1 \rangle$, hence the abelianization of G is a direct product of three copies of $\mathbb{Z}_p$. But the original pro-p presentation of G given by the generators x, y, z shows that the abelianization of G is $\mathbb{Z}/(p^s) \times \mathbb{Z}_p \times \mathbb{Z}_p$, a contradiction. Another option for D_1 is to have a pro-p presentation $\langle y_1, y_2 \mid y_1^{p^r}[y_1, y_2] = 1 \rangle$ then G has a pro-p presentation $\langle y_1, y_2, y_3 \mid y_1^{p^r}[y_1, y_2] = 1 \rangle$. By looking at the abelianization of G we get that $s = r$. But then looking at the maximal nilpotent quotient of class 2 of G we will show that the pro-p presentations given by generators x, y, z and y_1, y_2, y_3 cannot give isomorphic pro-p groups.

Indeed let N_1 be the maximal nilpotent quotient of class 2 of the pro-p group with presentation $\langle x, y, z \mid z^{p^s} = [x, y] \rangle$ and N_2 be the maximal nilpotent quotient of class 2 of the pro-p group with presentation $\langle y_1, y_2, y_3 \mid y_1^{p^r}[y_1, y_2] = 1\rangle$. Note that for a profinite group M which is nilpotent of class 2 and for $a, b, c \in M$ the commutator calculations used in Lemma 2 reduce to

$$[ab, c] = [a, c] \cdot [b, c], \ [a, bc] = [a, c] \cdot [a, b]$$

In particular, $[a, b]^{p^s} = [a^{p^s}, b]$. Furthermore, if M is (topologically) generated by a set S then $[M, M]$ is (topologically) generated by the set $[S, S] = \{[x, y] \mid x, y \in S\}$. In our case N_1 is (topologically) generated by $\{x, y, z\}$, hence $[N_1, N_1]$ is (topologically) generated by $[x, y], [x, z], [y, z]$. By the above calculations $[x, z]^{p^s} = [x, z^{p^s}] = [x, [x, y]] = 1$ and similarly $[y, z]^{p^s} = 1$. Hence if $a \in [N_1, N_1]$, then a^{p^s} is in the subgroup of $[N_1, N_1]$ (topologically) generated by $[x, y]^{p^s}$.

We claim that $[x, y]$ is of infinite order in N_1. Indeed let B be the pro-p group with finite presentation $\langle a, b \mid [[a, b], b] = 1, [[a, b], a] = 1\rangle$ and B_1 be the discrete group with presentation $\langle a_1, b_1 \mid [[a_1, b_1], b_1] = 1, [[a_1, b_1], a_1] = 1\rangle$. Then B_1 is a residually p-group, B is the pro-p completion of B_1 and the canonical map $\theta : B_1 \to B$ given by $\theta(a_1) = a, \theta(b_1) = b$ is injective. In particular as the order of $[a_1, b_1]$ is infinite, the order of $[a, b] = \theta([a_1, b_1])$ is infinite. Therefore B is a central extension of $\mathbb{Z}_p$ by $\mathbb{Z}_p \times \mathbb{Z}_p$. Then using the commutator calculations the specialization $x \to a^{p^s}$, $y \to b$, $z \to [a, b]$ extends to a homomorphism $\mu : N_1 \to B$. As $\mu([x, y]) = [a^{p^s}, b] = [a, b]^{p^s}$ is of infinite order, $[x, y]$ is of infinite order.

Note that in N_2 the image of y_1 has finite order : $1 = (y_1^{p^r}[y_1, y_2])^{p^r} = y_1^{p^{2r}}[y_1^{p^r}, y_2] = y_1^{p^{2r}}[[y_1, y_2]^{-1}, y_2] = y_1^{p^{2r}}$ in N_2. Thus there is an element of $N_2 \setminus [N_2, N_2]$ that is of finite order. We show that N_1 does not have this property. Assume that b is an element of $N_1 \setminus [N_1, N_1]$ of finite order. As $N_1/[N_1, N_1] \simeq \mathbb{Z}/(p^s) \times \mathbb{Z}_p \times \mathbb{Z}_p$ we have $b = z^k a$ for some $a \in [N_1, N_1] \subseteq Z(N_1)$ and some $0 < k < p^s$. Hence $b^{p^s} = z^{kp^s} a^{p^s} = [x, y]^k a^{p^s}$ is of finite order and as indicated above a^{p^s} is in the subgroup (topologically) generated by $[x, y]^{p^s}$. Then b^{p^s} is in the subgroup (topologically) generated by $[x, y]$, and as $[x, y]$ has infinite order this subgroup is isomorphic to $\mathbb{Z}_p$. Thus $1 = b^{p^s} = [x, y]^k a^{p^s}$ and $[x, y]^k$ is in the subgroup (topologically) generated by $[x, y]^{p^s}$, a contradiction.

2) If $q = 2$ then the minimal number of generators of D_1 is 2 or 3. In the latter case $G = D_1$ but by Theorem 4 G is not Demushkin. If the minimal number of generators of D_1 is 2, $N_1 = \mathbb{Z}_p$ and by Theorem 5 D_1 has a pro-p presentation $\langle y_1, y_2 \mid y_1^{2+2^f}[y_1, y_2]\rangle$ for some integer $f \geq 2$ or $f = \infty$.

Hence G has a pro-p presentation $\langle y_1, y_2, y_3 \mid y_1^{2+2^f}[y_1, y_2] = 1\rangle$. Looking at the abelianization of G we deduce that $2 + 2^f = 2^s$. As $f \geq 2$ or $f = \infty$ (where 2^∞ is defined as 0) we deduce that $s = 1, f = \infty$. Then we get two presentations of G as in the case 1 but for the specific values $p = 2, r = 1$. The same proof as in case 1 shows that the pro-p presentations given by generators x, y, z and y_1, y_2, y_3 cannot give isomorphic pro-p groups.

Acknowledgements

The authors thank the referee for the many suggestions that improved the paper.

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