Automorphisms of Pro-p groups of finite virtual cohomological dimension

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Abstract

Let G be a pro-p group of finite cohomological dimension and type FP_∞ and T is a finite p-group of automorphisms of G. We prove that the group of fixed points of T in G is again a pro-p group of type FP_∞ (in particular it is finitely presented). Moreover we prove that a pro-p group G of type FP_∞ and finite virtual cohomological dimension has finitely many conjugacy classes of finite subgroups.

Introduction

In the 80's Gersten proved the following remarkable theorem : the group of fixed points of an automorphism of a free group of finite rank is finitely generated [6]. This was a generalisation of an earlier result of Dyer and Scott [4] who proved that in the case of an automorphism of finite order the subgroup of fixed points is a free factor (the proof is based on Stalling's theory of ends). The pro-p version of Dyer-Scott's result for automorphisms of order p^n was obtained by Scheiderer [12] and later generalised to the infinitely generated case [9]. Note however that if the order of the automorphism is a

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natural number coprime to p and the original free pro-p group is not procyclic then the subgroup of fixed points is always (topologically) infinitely generated [7]. If the order of the automorphism of a free pro-p group of finite rank (as a super natural number) is p^∞ it was conjectured in [8] that the subgroup of fixed points is (topologically) finitely generated but this is still open.

In this note we extend Scheiderer's result to pro-p groups of finite cohomological p-dimension and type FP_∞. We state the main theorem for profinite groups and deduce in particular the pro-p case. Throughout the paper p denotes natural prime number.

Theorem 1. *Let H be a profinite group of finite cohomological p-dimension $cd_p(H) = d$ such that for every $i \geq 1$ the cohomology group $H^i(H, \mathbb{F}_p)$ is finite and σ be a continuous automorphism of H of order p. Then for the profinite group P of fixed points of σ we have*

$$\sum_{j\geq 0} dim H^j(P, \mathbb{F}_p) \leq \sum_{j\geq 0} dim H^j(H, \mathbb{F}_p)$$

Corollary 1. *Under the conditions of Theorem 1 if P is a free profinite group then P is (topologically) finitely generated.*

We recall that a pro-p group H is of homological type FP_∞ if the trivial pro-p $\mathbb{Z}_p[[H]]$-module $\mathbb{Z}_p$ has a pro-p projective resolution with all $\mathbb{Z}_p[[H]]$ modules finitely generated (topologically or abstractly is the same). This is equivalent with the Galois cohomology groups $H^j(H, \mathbb{F}_p)$ being finite for every $j \geq 0$.

Corollary 2. *Let H be a pro-p group of type FP_∞ and of finite cohomological dimension $cd_p(H) = d$ and T be a finite p-group of automorphisms of H. Then the group P of fixed points of T is a pro-p group of type FP_∞, in particular is finitely presented.*

Furthermore if H is a pro-p Poincare duality group of dimension 2, i.e. a Demushkin group, then P is a free pro-p group of rank at most $1 + dim H^1(H, \mathbb{F}_p)$.

Corollary 3. *Every pro-p group G of type FP_∞ and virtually finite cohomological dimension has finitely many conjugacy classes of finite subgroups and for every finite subgroup K of G the centraliser and the normaliser of K (in G) are again of type FP_∞.*

We note that Corollary 3 does not hold for discrete groups by the main result of [10] but a discrete version of Corollary 2 holds if we substitute the property FP_∞ with all cohomology groups $H^i(\ ,\mathbb{F}_p)$ being finite both in the assumptions and conclusions and in the case of Poincare duality group assume that the group is orientable. The first assertion can be proved by obvious induction on the order of T using the original Brown's theorem [2, X, 7.4], [3] exactly as in the proof of Corollary 2. The proof of the second assertion is similar to the proof of Corollary 2 and follows from the Strebel's result [14] plus the fact that the subgroup P cannot be of finite index. This can be seen by going to the pro-p completion $\widehat{H}_p$ of H that is a Poincare duality pro-p group [5, 5.12]. Indeed, since H is residually free and hence residually p [1, last line p. 424], H embeds in $\widehat{H}_p$; so we can extend σ to continuous automorphism $\widehat{\sigma}$ of $\widehat{H}_p$ and by the proof of Corollary 2 the fixed points of $\widehat{\sigma}$ cannot form a subgroup of finite index in $\widehat{H}$).

1 Proofs

1.1 Profinite case : proof of Theorem 1

The starting point is the following result.

Theorem 2. *[13, Cor. 12.19] Let G be a profinite group with no subgroup isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ and and assume that G has an open subgroup of finite cohomological p-dimension d. Then the restriction map*

$$H^n(G, A) \to \prod_S H^n(N_G(S), A) \tag{1}$$

is a dense embedding for $n > d$ and every finite discrete p-primary G-module A. Here the direct product is over a set of representatives S of the conjugacy classes of subgroups of G of order p.

Let H be a profinite group of finite cohomological p-dimension $cd_p(H) = d$ and σ be an automorphism of H of order p. Define $G = H \rtimes \langle \sigma \rangle$. By the Hochschild-Serre spectral sequence

$$dim H^n(G, \mathbb{F}_p) \leq \sum_{i+j=n} dim H^i(G/H, H^j(H, \mathbb{F}_p)) \tag{2}$$

where dim is the dimension as a vector space over $\mathbb{F}_p$. As $G/H \simeq \mathbb{Z}/p\mathbb{Z}$ the trivial $\mathbb{Z}_p[G/H]$-module $\mathbb{Z}_p$ has a free resolution $\mathcal{F}$ with all free modules being of rank 1. Hence for every discrete $\mathbb{Z}_p[G/H]$-module M the complex $Hom_{\mathbb{Z}_p[G/H]}(\mathcal{F}^{del}, M)$ contains only modules isomorphic to M where upper index del denotes the deleted complex. In particular for $M = H^j(H, \mathbb{F}_p)$ we get $dim H^i(G/H, H^j(H, \mathbb{F}_p)) \leq dim H^j(H, \mathbb{F}_p)$. The last combined with (2) gives

$$dim H^n(G, \mathbb{F}_p) \leq \sum_{0 \leq j \leq n} dim H^j(H, \mathbb{F}_p) < \infty \tag{3}$$

Then for every n we have $dim H^n(G, \mathbb{F}_p) < \infty$ and for $n > d$ the dense embedding (1) is an isomorphism and the direct product in (1) is finite, hence

G has only finitely many conjugacy classes of subgroups of order p. (4)

Note that for a subgroup S of G of order p we have $N_G(S) = C_G(S) = C_H(S) \times S$ and by Künneth formula

$$H^n(N_G(S), \mathbb{F}_p) = \oplus_{i+j=n} H^i(C_H(S), \mathbb{F}_p) \otimes_{\mathbb{F}_p} H^j(S, \mathbb{F}_p) = \oplus_{0 \leq i \leq n} H^i(C_H(S), \mathbb{F}_p)$$

Then

$$dim H^n(N_G(S), \mathbb{F}_p) = \sum_{0 \leq i \leq n} dim H^i(C_H(S), \mathbb{F}_p). \tag{5}$$

Applying (5) for $n > d$ and combining with (1) we get that

$$\sum_S \sum_{0 \leq i \leq n} dim H^i(C_H(S), \mathbb{F}_p) = \sum_S dim H^n(N_G(S), \mathbb{F}_p)$$

$$= dim H^n(G, \mathbb{F}_p) \leq \sum_{0 \leq i \leq n} dim H^i(H, \mathbb{F}_p) \tag{6}$$

where S runs over representatives of the conjugacy classes of the subgroups of p-order in G. For S the subgroup generated by σ we have that $C_G(S)$ is the group P of the fixed points of σ in H and (6) finishes the proof of Theorem 1.

2 Pro-p groups : proof of Corollary 2 and Corollary 3

Lemma 1. *Let H be a torsion free pro-p group and T a finite p-group of automorphisms of H. Then the subgroup P of fixed points of T has infinite index in H.*

Proof. Clearly, it is sufficient to show the lemma for T of order p. Suppose that P has finite index in H. Then there is a normal open subgroup N in H such that $N \subseteq P$ and N is maximal (under inclusion) with this properties. The group $M = G/N \rtimes T$ is a finite p-group with the action of T on G/N induced by the T-action on G. Thus M is nilpotent and so there is a non-trivial element $gN \in Z(M) \cap G/N$. Let σ be a generator of T. Then $\sigma(gN) = gN$, hence $\sigma(g) = gh$ for some $h \in N \subseteq P$. In particular $g = \sigma^p(g) = gh^p$, so $h^p = 1$ and as H is torsion-free $h = 1$. Then $g \in P$ and the subgroup of H generated by N and g contradicts the maximality of N.

□

Proof of Corollary 2. We use induction on the order of T. Let σ be a central element of T of order p. Then by Theorem 1 the group of fixed points S_0 of σ is of type FP_∞, obviously it is of finite cohomological dimension. Now the quotient group T_1 of T by the subgroup generated by σ is a group of automorphisms of S_0. By induction the group of fixed points of T_1 on S_0 is of homological type FP_∞.

Suppose now that H is a pro-p Poincare duality group of dimension 2. By the preceding lemma the subgroup P of fixed points of T cannot have finite index in H, hence by [11, Exer. 3,p. 174] it is of finite cohomological dimension at most 1, so is a free pro-p group.

We note that by duality $H^2(H, \mathbb{F}_p) \simeq H_0(H, \mathbb{F}_p^*)$ where $\mathbb{F}_p^* \simeq D \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{F}_p \simeq \mathbb{F}_p$ as pro-p groups since $D = \mathbb{Z}_p$ is the dualising module. As $Aut(\mathbb{F}_p^*)$ has order $p-1$ the only possible action of H on $\mathbb{F}_p$ is the trivial one and we get $H_0(H, \mathbb{F}_p^*) \simeq \mathbb{F}_p$. In particular by Theorem 1

$$1 + dim H^1(P, \mathbb{F}_p) \leq \sum_{0 \leq i \leq 2} dim H^i(H, \mathbb{F}_p) = 2 + dim H^1(H, \mathbb{F}_p)$$

so the rank of P is at most $1 + dim H^1(H, \mathbb{F}_p)$ as required.

Proof of Corollary 3. We use the ideas of the proof of [12, Cor. 1.3]. Let H be a normal subgroup of G such that H has finite cohomological dimension and let H be maximal (under inclusion) with this property. Then G/H is a finite p-group. Let K be a finite subgroup of G. Then $H.K = H \rtimes K$ is a pro-p subgroup of finite index in G, so is of homological type FP_∞. Then without loss of generality we can assume that $G = H \rtimes K$. Let p^n be the order of the group K. If $n = 1$ Corollary 3 follows from (4) and Theorem 1. Note that for general n, $C_G(K) \cap H = N_G(K) \cap H$ and by Corollary 2 $C_G(K) \cap H$ is of type FP_∞, hence $C_G(K)$ is of type FP_∞ and obviously $C_G(K)$ is virtually of finite cohomological dimension. We prove by induction on s that up to conjugation G has only finitely many subgroups of order p^s. If $s = 1$ and t is an element of G of order p it suffices to show that the group $H \rtimes \langle t \rangle$ has only finitely many conjugacy classes of groups of order p (this is (4)). For the inductive step it suffices to show that for a fixed subgroup T of G of order p^{s-1} the subgroups S of G of order p^s containing T lie in finitely many G-conjugacy classes. As $S \subseteq N_G(T)$ it is sufficient to show that the pro-p group $N_G(T)/T$ has finitely many orbits (under conjugation) of elements of order p. Note that as T is finite and $N_G(T)$ is of type FP_∞ the pro-p group $N_G(T)/T$ is of type FP_∞ and obviously $N_G(T)/T$ is virtually of finite cohomological dimension as $N_G(T)$ is virtually of finite cohomological dimension. Note that for $N_G(T)/T$ the case $s = 1$ (that was already proved) applies.

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